Math 323 - 232 Second Major Exam March 21, 2024

Name: ______________________________ ID #: ______________.

Q1) a) State Lagrange's Theorem.

b) Prove Lagrange's Theorem.

c) Suppose that G is an Abelian group with an odd number of elements. Show that the product of all of the elements of G is the identity.

Q2) a) Show that $G \oplus H$ is Abelian if and only if G and H are Abelian.

b) Find a subgroup of $\mathbb{Z}_{12} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{15}$ that has order 9.

Q3) a) Prove that a subgroup H of G is normal in G if and only if $xHx^{-1} \subseteq H$ for all x in G.

b) Let p be a prime. Show that if H is a subgroup of a group of order $2p$ that is not normal, then H has order 2.

Q4) a) Let ϕ be a homomorphism from a group G to a group $\bar{G}$. If $\overline{K}$ is a normal subgroup of $\bar{G}$, then $\phi^{-1}(\overline{K}) = \{k \in G \mid \phi(\mathrm{k}) \in \overline{K}\}$ is a normal subgroup of G.

b) Prove that there is no homomorphism from A_4 onto $\mathbb{Z}_2$.

Q5) a) Suppose that G is an Abelian group of order 120 and that G has exactly three elements of order 2. Determine the isomorphism class of G.

b) Determine the isomorphism class of $Aut(\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5)$.

Q6) Prove or disprove:

a) If $K \lhd H$ and $H \lhd G$, then $K \lhd G$.

b) $U(40) \oplus \mathbb{Z}_6 \cong U(72) \oplus \mathbb{Z}_4$.

c) The mapping $\emptyset: \mathbb{Z}_{12} \to \mathbb{Z}_{10}$ defined by $\emptyset(\mathrm{x}) = 3x$ is a homomorphism.