Math 323 - 232 Final Exam May 29, 2024

Name: ____ ID #: __________________.

Q1) a) Prove that a group with two elements of order 2 that commute must have a subgroup order 4.

b) Determine the subgroup lattice for $\mathbb{Z}_{84}$.

c) How many elements of order 5 are in S_7. How many elements of order 4 are in S_6. Explain your answer.

Q2) a) Let G be a group with $|G| = 33$. What are the possible orders for the elements of G? Show that G must have an element of order 3.

b) Let G be a group of order 30 and let H a non-cyclic subgroup of G with odd order. Show that H is a normal subgroup of G.

c) If H is a normal subgroup of G and $|H| = 2$, prove that H is contained in the center of G.

Q3) a) Find all Abelian groups (up to isomorphism) of order 360.

b) Let $R = \{0,2,4,6,8\}$ under addition and multiplication modulo 10. Porve that R is a field.

Q4) Let $R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \middle| \, a, b \in \mathbb{Z} \right\}$, and let φ be a mapping that takes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ to $a - b$.

a) Show that the mapping φ is a ring homomorphism.

b) Determine the Kernal of φ.

c) Show that $R/\mathrm{Ker}\,\varphi$ is isomorphic to $\mathbb{Z}$.

d) Is $\mathrm{Ker}\,\varphi$ a prime ideal? Explain your answer.

e) Is $\mathrm{Ker}\,\varphi$ a maximal ideal? Explain your answer.

Q5) a) Prove that the ideal $\langle x \rangle$ in $\mathbb{Q}[x]$ is maximal.

b) Suppose that $f(x) \in \mathbb{Z}_p[x]$ and $f(x)$ is irreducible over $\mathbb{Z}_p$, where p is a prime. If $\deg f(x) = n$, prove that $\mathbb{Z}_p[x]/\langle f(x) \rangle$ is a field with p^n elements.

c) Construct a field of order 25.

Q6) a) Prove that in an integral domain, every prime is an irreducible.

b) Give an example of an integral domain in which there are elements that are irreducible but not primes. Explain your answer. Is this integral domain a principal ideal domain?