King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 323 (Term 242)

Major Exam 1 (100 minutes - 5 Problems)

Name : ______________________ **ID :** ______________

Problem 1. Let A_n denote the set of all *even* permutations of degree $n \geq 3$.

- **(1)** Show that A_n is a subgroup of the symmetric group S_n and prove that $|A_n| = \dfrac{n!}{2}$.
- **(2)** Let $\sigma \in A_n$ such that $\sigma = (r_1\text{-cycle})(r_2\text{-cycle})\ldots(r_k\text{-cycle})$, where the r_i-cycles are non-trivial disjoint with $r_1 + r_2 + \cdots + r_k = n$. Show that n and k have the same parity (i.e., both are either even or odd).
- **(3)** If $n = 8$, find all possible orders of σ.
- **(4)** If $n = 12$ and $k = 4$, find the minimum and maximum possible orders of σ.

Problem 2. Let G be a group of order $n = pqr$, where p, q, r are prime numbers with $p < q < r$.

- **(1)** Let H be a subgroup of G such that H contains two non-identity elements with distinct orders. Find the smallest possible value for the order of H.

  Throughout, let $G = \dfrac{\mathbb{Z}}{78\mathbb{Z}}$.
- **(2)** Draw the lattice of all subgroups of G and, for each subgroup, give its generator and order.
- **(3)** Determine explicitly the subgroup H described in (1), and find all $\overline{a} \in H$ such that $H = \langle \overline{a} \rangle$.

Problem 3. Let $G = U(33)$ be the multiplication group modulo 33.

- **(1)** Show $\left|2\right| = 10$ and find two elements of order 2.
- **(2)** Determine the isomorphism class of G.
- **(3)** Express G as an internal direct product of two cyclic subgroups.

Problem 4. Let G be a non-cyclic group of order 6. Prove the following assertions:

- **(1)** $\forall\, x \in G \setminus \left\{1\right\}$, $\left|x\right| = 2$ or 3.
- **(2)** There exists $x \in G$ such that $\left|x\right| = 3$.
- **(3)** There exists $y \in G$ such that $\left|y\right| = 2$.
- **(4)** G is isomorphic to D_3, the Dihedral group of degree 3 (and order 6).
- **(5)** D_3 is the smallest non-abelian group.

Problem 5. Let G be a *finite* group, and let $a, b \in G$. We say that a and b are **conjugate** in G if there exists $x \in G$ such that $b = xax^{-1}$. Conjugacy defines an **equivalence relation** on G, and the **conjugacy class** of an element $a \in G$ is the **set** : $\widehat{a} = \left\{xax^{-1} \mid x \in G\right\}$.

- **(1)** Let $a \in G$ and let $H_a := \left\{x \in G \mid xa = ax\right\}$ be the centralizer of a (which is subgroup of G). Show that $\left|G : H_a\right| = \left|\widehat{a}\right|$.
- **(2)** Show that $\left|G\right| = \sum_{a} \left|G : H_a\right|$, where the sum ranges over *one representative* a of the class $\widehat{a}$
- **(3)** Show that $\left|G\right| = \left|Z(G)\right| + \sum_{a \notin Z(G)} \left|G : H_a\right|$, where $Z(G)$ denotes the center of G.
- **(4)** Assume that $\left|G\right| = p^n$, where p is a prime number and $n \geq 1$.
  - (a) Show that p divides $\left|G : H_a\right|$, for each $a \notin Z(G)$.
  - (b) Show that $Z(G)$ is not trivial.