King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 323 (Term 242)
Major Exam 2 (100 minutes - 5 Problems)

Problem 1. **[20]** Let R be an integral domain and consider the subring of $M_2(R)$ given by

$$A := \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in R \right\}.$$

(1) Show that A is isomorphic to $R \times R$ as groups.
(2) Show that A is NOT isomorphic to $R \times R$ as rings.
(3) Show that A is isomorphic to $\dfrac{R[X]}{(X^2)}$ as rings.
(4) Assume $R = \dfrac{\mathbb{Z}}{p\mathbb{Z}}$, where p is a prime number, and let $\mathrm{Nil}(A)$ and $\mathrm{Idem}(A)$ denote the sets of all nilpotent and idempotent elements of A, respectively. Find $\left|\mathrm{Nil}(A)\right|$ and $\left|\mathrm{Idem}(A)\right|$.

Problem 2. **[20]** Let $p \lneqq q$ be prime numbers.

(1) Show that the polynomial ring $\dfrac{\mathbb{Z}}{p^2\mathbb{Z}}\left[X\right]$ has infinitely many units.
(2) Solve the equation $x^q = 1$ in $\dfrac{\mathbb{Z}}{pq\mathbb{Z}}$.
(3) Find a root for the polynomial $f := X^p - q$ over $\dfrac{\mathbb{Z}}{pq\mathbb{Z}}$.
(4) Prove that $\dfrac{\mathbb{Z}}{pq\mathbb{Z}} \cong \dfrac{\mathbb{Z}}{p\mathbb{Z}} \times \dfrac{\mathbb{Z}}{q\mathbb{Z}}$ as rings.

Problem 3. .

(1) Show that a positive integer is divisible by 9 if and only if the sum of its digits is divisible by 9.
(2) Use (1) to find the smallest 20-digit number that is divisible by 9?
(3) Show that a positive integer is divisible by 11 if and only if the alternating sum of its digits is divisible by 11.
(4) Use (3) to find the smallest 20-digit number that is divisible by 11.

Problem 4. **[20]** Let A be a commutative ring such that every proper ideal is contained in a maximal ideal. Let N denote the intersection of all prime ideals of A and J denote the intersection of all maximal ideals of A.

(1) Let x be a nilpotent element of A. Show that $1 + x$ is a unit of A, and deduce that the sum of a nilpotent element and a unit is a unit.
(2) Show that $x \in J$ if and only if $1 - ax$ is a unit for all $a \in A$.
(3) Assume that every ideal not contained in N contains a non-zero idempotent. Prove that $N = J$.
(4) Assume that every $x \in A$ satisfies $x^n = x$, for some $n \geq 2$. Prove that every prime ideal is maximal.

Problem 5. **[20]** Consider the following polynomials in $\mathbb{Q}[X]$:

(1) $f_1 = X^3 + 2X^2 + X - 1$
(2) $f_2 = 2X^3 + X^2 + 3X + 2$
(3) $f_3 = X^4 + 4X^2 + 6$
(4) $f_4 = 2X^4 + 4X^2 + 2$
(5) $f_5 = X^5 - X^3 + 3X^2 - 3$

Determine which of these polynomials is irreducible or reducible over $\mathbb{Q}$, and explain your reasoning.