King Fahd University of Petroleum and Minerals
Department of Mathematics

Math 323 (Term 242)

Final Exam (150 minutes - 20 Exercises)

Code 1

Exercise 1. Let n be a positive integer with decimal representation $a_k a_{k-1} \dots a_1 a_0$. Then, in the ring $\dfrac{\mathbb{Z}}{16\mathbb{Z}}$, n is equal to:

(a) $a_0 + 2a_1 - 8a_3$
(b) $8a_{k-3} - 4a_{k-2} + 2a_{k-1} - a_k$
(c) $a_0 - 10a_1 + 4a_2$
(d) $a_0 + 10a_1 + 4a_2 + 8a_3$
(e) $a_k + 10a_{k-1} + 4a_{k-2} + 8a_{k-3}$

Exercise 2. Let $\left\{\overline{s_1}, \overline{s_2}, \dots, \overline{s_n}\right\}$ denote the set of all distinct solutions of the equation $x^{42} = 1 \pmod{67}$, where $\mathbf{1} \le s_1 < s_2 < \cdots < s_n \le \mathbf{66}$. Then, $s_1 + s_2 + \cdots + s_n =$

(a) 67
(b) 134
(c) 201
(d) 268
(e) 402

Exercise 3. The number of all irreducible monic quadratics in $\dfrac{\mathbb{Z}}{7\mathbb{Z}}[X]$ is equal to

(a) 21
(b) 7
(c) 0
(d) 28
(e) 49

Exercise 4. In the group $\dfrac{\mathbb{Z}}{9\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{3\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{3\mathbb{Z}}$, the number of elements of order 9 is equal to

(a) 18
(b) 24
(c) 36
(d) 48
(e) 54

Exercise 5. Let G be an abelian group of order 24, which has an element of order 12 and two elements of order 2. Then, G is isomorphic to:

(a) $\dfrac{\mathbb{Z}}{6\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{4\mathbb{Z}}$
(b) $\dfrac{\mathbb{Z}}{8\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{3\mathbb{Z}}$
(c) $\dfrac{\mathbb{Z}}{12\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{3\mathbb{Z}}$
(d) $\dfrac{\mathbb{Z}}{12\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{12\mathbb{Z}}$
(e) $\dfrac{\mathbb{Z}}{2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{3\mathbb{Z}}$

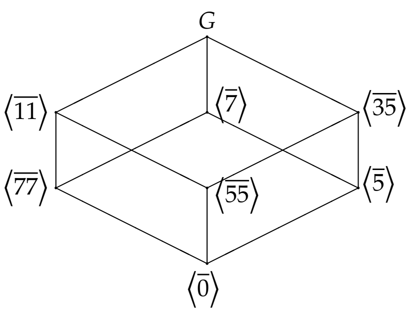
Exercise 6. Let $G := U(22) = \left\{1 \leq k \leq 21 \mid (k,22) = 1\right\}$ be the group under multiplication modulo 22. In G, we have $3^5 = 5^5 = 9^5 = 1$, $7^2 = 5$, $13^2 = 15$, $17^2 = 3$, $21^2 = 1$, and $7^5 = 13^5 = 17^5 = 21$. Then

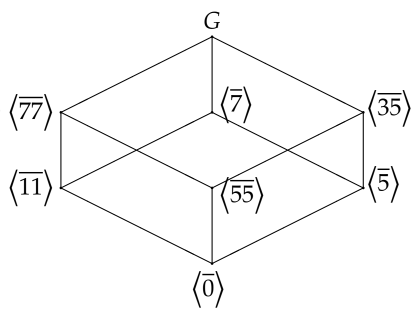
(a) $G = \left\langle 9 \right\rangle$
(b) $G = \left\langle 17 \right\rangle$
(c) $G \cong \left\langle 3 \right\rangle \times \left\langle 5 \right\rangle$
(d) $G \cong \left\langle 3 \right\rangle \times \left\langle 7 \right\rangle$
(e) $G \cong \left\langle 13 \right\rangle \times \left\langle 21 \right\rangle$

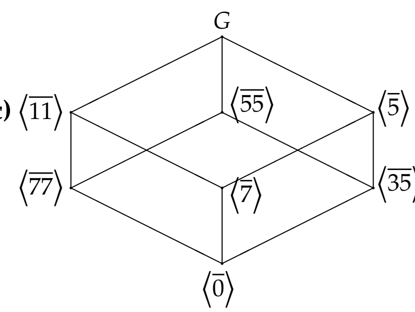
Exercise 7. Let $G := \{1, 4, 11, 14, 16, 19, 26, 29, 31, 34, 41, 44\}$ be a group under multiplication modulo 45. Then:

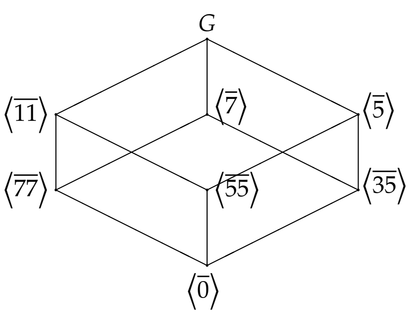
(a) G is the internal direct product of $\langle 11 \rangle$ and $\langle 16 \rangle$
(b) G is the internal direct product of $\langle 11 \rangle$ and $\langle 26 \rangle$
(c) G is the internal direct product of $\langle 16 \rangle$ and $\langle 44 \rangle$
(d) G is the internal direct product of $\langle 41 \rangle$ and $\langle 26 \rangle$
(e) G is the internal direct product of $\langle 41 \rangle$ and $\langle 44 \rangle$

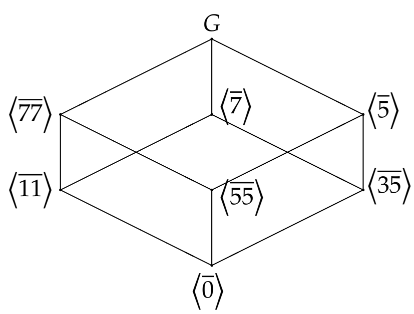
Exercise 8. The *lattice* of all subgroups of $\dfrac{\mathbb{Z}}{385\mathbb{Z}}$ is.

(a)


(b)


(c)


(d)


(e)


Exercise 9. Let $p < q$ be two primes and G a finite group of order p^2q. Let H be a normal subgroup of G of order p and let $c(H) := \left\{g \in G \mid ghg^{-1} = h\ ,\ \forall h \in H\right\}$ the centralizer of H in G (which is a normal subgroup of G). Then:

(a) $\left|c(H)\right| = p$
(b) $\left|c(H)\right| = q$
(c) $\left|c(H)\right| = p^2$
(d) $\left|c(H)\right| = pq$
(e) $\left|c(H)\right| = p^2q$

Exercise 10. In $\dfrac{\mathbb{Z}}{11\mathbb{Z}}\left[X\right]$, the remainder of the division of X^{53} by $X+3$ is equal to:

(a) 2
(b) 10
(c) 4
(d) 8
(e) 6

Exercise 11. Let A_8 denote the alternating group of degree 8 and let $\sigma \in A_8$ with $\sigma = \alpha_1\alpha_2\ldots\alpha_k$, where the α_i's are non-trivial **disjoint** r_i-cycles ($r_i \geq 2$) and $r_1 + r_2 + \cdots + r_k = 8$. Let a and b denote the maximum order and minimum order, respectively, σ can have. Then, $a + b =$

(a) 9
(b) 10
(c) 17
(d) 21
(e) 19

Exercise 12. Let σ be an *odd* permutation of degree 10 such that $\sigma = \alpha_1\alpha_2\ldots\alpha_k$, where the α_i's are non-trivial disjoint r_i-cycles ($r_i \geq 2$) and $r_1 + r_2 + \cdots + r_k = 10$. Then, all possible values for k are:

(a) 3, 5
(b) 1, 3, 5
(c) 1, 2, 3, 4, 5
(d) 2, 4, 5
(e) 2, 4

Exercise 13. Let A_{11} denote the alternating group of degree 11 and let $\sigma \in A_{11}$ with $\sigma = \alpha_1\alpha_2\alpha_3$, where the α_i's are non-trivial disjoint r_i-cycles ($r_i \geq 2$) and $r_1 + r_2 + r_3 = 11$. If $\left|\alpha_1\right| \lneqq \left|\alpha_2\right| \lneqq \left|\alpha_3\right|$, then $\left|\sigma\right| =$

(a) 1 or 11
(b) 6 or 20
(c) 6 or 14
(d) 12 or 15
(e) 14 or 20

Exercise 14. Let G be a group of order 385. If $x \in G \setminus \left\{1\right\}$ such that $\left|x\right|$ is **not prime** and two of x^5, x^7, x^{40} are equal, then $\left|x^{42}\right| =$

(a) 5
(b) 7
(c) 11
(d) 35
(e) 77

Exercise 15. Let $G := \dfrac{\mathbb{Z}}{11\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{11\mathbb{Z}}$ and let n denote the number of subgroups of G of order 11. Then

(a) $n = 1$
(b) $n = 10$
(c) $n = 11$
(d) n = 12
(e) $n = 120$

Exercise 16. Let $f := X^p + p^2X + p$ and $g := X^{p+1} - X$, where p is a prime number. Consider the ring $R := \dfrac{\mathbb{Q}[X]}{\left(\phi^2\right)}$, where $\phi := 1 + X + X^2 + \cdots + X^{p-1}$.

(a) $\overline{f}$ and $\overline{g}$ are units in R
(b) $\overline{f}$ and $\overline{g}$ are nilpotent in R
(c) $\overline{f}$ is nilpotent and $\overline{g}$ is a unit in R
(d) $\overline{f}$ is a unit and $\overline{g}$ is nilpotent in R
(e) None of these statements are true

Exercise 17. The ring $\dfrac{\mathbb{Q}[X]}{(X^4 - 12X^2 + 27)}$ is isomorphic to:

(a) $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$
(b) $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}(\sqrt{3})$
(c) $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}(i)$
(d) $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$
(e) $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}(\sqrt{3}) \times \mathbb{Q}(\sqrt{3})$

Exercise 18. The polynomial $f := X^2 + 1$ is irreducible in

(a) $\dfrac{\mathbb{Z}}{5\mathbb{Z}}[X]$

(b) $\dfrac{\mathbb{Z}}{17\mathbb{Z}}[X]$

(c) $\dfrac{\mathbb{Z}}{31\mathbb{Z}}[X]$

(d) $\dfrac{\mathbb{Z}}{37\mathbb{Z}}[X]$

(e) $\dfrac{\mathbb{Z}}{101\mathbb{Z}}[X]$

Exercise 19. Which one of the following statements is CORRECT?

(a) $\dfrac{\mathbb{Z}}{3\mathbb{Z}}[i]$ is NOT a field

(b) $\dfrac{\mathbb{Z}}{5\mathbb{Z}}[i]$ is an integral domain

(c) $\dfrac{\mathbb{Z}}{11\mathbb{Z}}[i]$ is an infinite ring

(d) $\dfrac{\mathbb{Z}}{13\mathbb{Z}}[i]$ is a field

(e) $\dfrac{\mathbb{Z}}{17\mathbb{Z}}[i]$ is NOT a field

Exercise 20. The polynomial $f := X^4 + 1$ is irreducible over

(a) $\dfrac{\mathbb{Z}}{2\mathbb{Z}}$

(b) $\mathbb{R}$

(c) $\mathbb{Q}[\sqrt{2}]$

(d) $\mathbb{Q}[i]$

(e) f is NOT irreducible over all these fields