King Fahd University of Petroleum and Minerals,

Department of Mathematics- Term 211

Major Exam 1: Math 325, Linear Algebra

Duration: 2 Hours

NAME :

ID :

Solve the following Exercises.

Exercise 1. (5-5-5-5 points)

Let $\mathbb{Q}$ be the field of rational numbers, $p < q$ be positive prime integers and set $K = \mathbb{Q}(\sqrt{p}) = \{a + b\sqrt{p} | a, b \in \mathbb{Q}\}$ and $V = K(\sqrt{q}) = \{x + y\sqrt{q} | x, y \in K\}$.

(1) Prove that K is a field.

(2) Prove that V is a vector space over K.

(3) Find a basis of V as a vector space over K and $dim_K V$.

(5) Find a basis of V as a vector space over $\mathbb{Q}$ and $dim_{\mathbb{Q}} V$.

Exercise 2. (5-5-5 points)

Let V be the real vector space given by $V = \mathbb{R}^3$. Which one of the subsets of V is a subspace of V. Justify for each one.

(1) $W_1 = \{(x, y, z) \in V | x + 2y + 4z = 0\}$.

(2) $W_2 = \{(x, y, z) \in V | x - y = 0, \text{ or } x + z = 0\}$.

(3) $W_3 = \mathcal{P}_x \cup \mathcal{P}_z$ where $\mathcal{P}_x = \{(0, y, z) | y, z \in \mathbb{R}\}$ and $\mathcal{P}_z = \{(x, y, 0) | x, y \in \mathbb{R}\}$.

Exercise 3. (7-5-7-6)

Let V be a vector space over a field K (not necessarily of finite dimension) and U, W and W' subspaces of V such that $V = U \bigoplus W = U \bigoplus W'$.

(1) Prove that W and W' are isomorphic.[find an isomorphism between W and W'].

(2) Assume that $V = \mathbb{R}^4$ and $U = span\{(1,0,1,0),(1,-1,1,-1),(0,1,0,1)\}$. Find a complement W of U.

Assume that V is the real vector space of all $n\times n$ matrices and $U = \{A \in V | A^t = A\}$ and $W = \{A \in V | A^t = -A\}$.

(3) Show that U and W are subspaces of V and $V = U \bigoplus W$.

(4) Find bases of U and W, and $dimU$ and $dimW$.

Exercise 4. (5-5-5-5-5 points)

Let V be the real vector space given by $V = \mathbb{R}^3$ and $f : V \longrightarrow V$ defined by $f(x, y, z) = (-x + y + z, x - y + z, x - y - z)$.

(1) Prove that f is a linear map on V.

(2) Find $Ker(f)$. Is f a one-to-one linear map? Justify.

(3) Find a basis of $ker(f)$ and $dim(ker(f))$.

(4) Find $Im(f)$, a basis of $Im(f)$ and $dim(Im(f))$.

(5) Is $V = ker(f) \bigoplus Im(f)$? Justify.

Exercise 5. (7-6-7)

(1) Find explicitly a linear map $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ such that $T(1,1,0) = (1,1)$ and $T(0,1,1) = (1,-1)$.

(2) Find the matrix representing T in the standard bases $\mathcal{S}_1, \mathcal{S}_2$ of $\mathbb{R}^3$ and $\mathbb{R}^2$.

(3) Let $\mathcal{B}_1 = \{(-1,1,0),(1,0,1),(0,0,-1)\}$ and $\mathcal{B}_2 = \{(2,1),(1,2)\}$. Find the matrix representing T in the bases $\mathcal{B}_1$ and $\mathcal{B}_2$.