King Fahd University of Petroleum and Minerals,

Department of Mathematics and Statistics- Term 211

Major Exam 2: Math 325, Linear Algebra

Duration: 150 minutes

NAME :

ID :

Solve the following Exercises.

Exercise 1. (5-5 points)
Let W be the subspace of $V=\mathbb{R}^3$ given by $W=span\{(1,2,1),(1,1,1)\}$ and consider the scalar product on V defined by $(x_1,x_2,x_3)|(y_1,y_2,y_3)=x_1y_1-x_2y_2-x_3y_3$.
(1) Find an orthonormal basis of W.
(2) Set $V=\mathbb{C}^2$ as a vector space over $\mathbb{C}$ with the product scalar defined by:
$(u|v)=x_1y_1-ix_2y_1-ix_1y_2-2x_2y_2$, $u=(x_1,x_2)$ and $v=(y_1,y_2)$. Find an othogonal basis of V.

Exercise 2. (5-5-5-5 points)

Let V be an n-dimensional vector space over a field K with a non-degenerate scalar product $(|)$, W a subspace of V and V^* its dual space.

(1) Prove that $dim V = dim(W) + dim W^{\perp}$.

(2) Prove that for any linear functional $T \in V^*$, there is a unique $w \in V$ such that $Tv = (v|w)$ for every $v \in V$.

Assume that $V = \mathbb{R}^3$ with the scalar product $(X|Y) = x_1y_1 + 2x_2y_2 + 4x_3y_3$, and let $B = \{v_1 = (0,1,0), v_2 = (1,0,1), v_3 = (0,1,1)\}$ and B^* its dual basis.

(3) Express $v_i^*(x,y,z)$ for each $i = 1,2,3$.

(4) Let $T : \mathbb{R}^3 \mapsto \mathbb{R}$ defined by $T(x,y,z) = x + y - z$. Find $w \in \mathbb{R}^3$ such that $Tv = (v|w)$ for every $v \in \mathbb{R}^3$.

Exercise 3. (5-5)

(1) Find the bilinear form associated to the quadratic form $q(x, y, z) = 4x^2 + 8xy + 2xz$.

(2) Find the quadratic form associated to the bilinear form:

$f(X, Y) = x_1y_1 - x_1y_2 + x_1y_3 - x_2y_1 + x_2y_3 - x_3y_1$.

Exercise 4. (8-5-2 points)

Let V be an n-dimensional ($n \geq 1$) vector space over $\mathbb{R}$ with a scalar product.

(1) Prove that there is an integer r such that for every orthogonal basis $B = \{v_1, \ldots, v_n\}$ of V, there are exactly r integers i such that $(v_i|v_i) > 0$.

Assume that $V = \mathbb{R}^3$ with the scalar product defined by:

$(X|Y) = x_1y_1 + x_1y_2 + x_2y_1 + x_2y_3 + x_3y_2$.

(1) Find an orthogonal basis of V with respect to the defined scalar product.

(2) Find the index of positivity.

Exercise 5. (5-5-5)

Let V and W be finite-dimensional vector spaces over $\mathbb{R}$ both having inner products and let $L : V \longrightarrow W$ be a linear map.

(1) Show that $Im(L)^{\perp} = Ker(L^*)$, where L^* is the adjoint of L.

(2) Show that $dim(Im(L)) = dim(Im(L^*))$.

(3) Show that the equation $Lx = y$ has a solution if and only if y is orthogonal to the kernel of L^*.

Exercise 6. (5-5-5-5)

Let V be an n-dimensional vector space over $\mathbb{R}$ with a positive definite scalar product $(|)$, T be a linear operator on V and T^* the operator adjoint of T.

(1) Show that both T^*T and TT^* are self-adjoint.

(2) Show that $ker(T) = ker(T^*T)$.

(3) If T is one-to-one, show that T^* and T^*T are invertible and find their inverses.

(4) Show that $Im(TT^*) = (ker(TT^*)^\perp$

Exercise 7. (5-5)

Let $V = \mathbb{R}^n$, T a unitary operator on V and A be matrix representing T in a basis B of V. (1) Find $det(A)$.

(2) Assume that T is annihilated by the polynomial $f(X) = X^2 - 1$. Is T a symmetric operator? Justify.