King Fahd University of Petroleum and Minerals,

Department of Mathematics- Term 211

Final Exam: Math 325, Linear Algebra

Duration: 3 Hours

NAME :

ID :

Solve the following Exercises.

Exercise 1. (6-6-6-6-5-6) points)

Let V be a finite-dimensional vector space over $\mathbb{K}$ with a positive definite scalar product and let T be a linear operator on V.

Part I. Assume that $K = \mathbb{C}$.

(1) Prove that for every $x, y \in V$,

$(Tx|y) = \frac{1}{4}[(T(x+y)|x+y) - (T(x-y)|x-y) + i(T(x+iy)|x+iy) - i(T(x-iy)|x-iy)]$.

(2) Prove that if $(Tx|x) = 0$ for every $x \in V$, then $T = 0$.

(3) Prove that T is self-adjoint if and only if $(Tx|x)$ is real for every $x \in V$.

Part II. Assume that $K = \mathbb{R}$ and suppose that $(Tv|v) = 0$ for every $v \in V$.

(4) Prove that $A + A^t = 0$.

(5) Prove that if A is symmetric, then $A = 0$.

(6) Find an example of a real finite-dimensional vector space V with a linear operator A such that $(Ax|x) = 0$ for every $x \in V$ but $A \neq 0$.

Exercise 2. (6-8-8-8)

Let V be an n-dimensional vector space over $\mathbb{C}$ with a definite positive scalar product and T a linear operator such that $TT^* = T^*T$, where T^* is the adjoint of T.

(1) Prove that $Nullspace(T) = Nullspace(T^*)$.

(2) Prove that $Nullspace(T)$ is the orthogonal complement of $range(T)$ (that is, $Nullspace(T) = (rang(T))^{\perp}$. Deduce that $Nullspace(T) = Nullspace(T^2)$.

(3) Suppose that there is two polynomials $f(X)$ and $g(X)$ relatively prime and $\alpha, \beta \in V$ such that $f(T)\alpha = g(T)\beta = 0$. Prove that $(\alpha|\beta) = 0$.

(4) Prove that there exist two linear self-adjoint operators T_1 and T_2 with $T_1T_2 = T_2T_1$ such that $T = T_1 + iT_2$ (i is the complex number with $i^2 = -1$).

Exercise 3. (10-6-9)

Let V be an n-dimensional vector space over the real field $\mathbb{R}$ with a positive definite scalar product $(|)$ and let T be a unitary operator on V.

(1) Prove that V has a direct sum decomposition $V = V_1 \bigoplus \cdots \bigoplus V_r$ where V_i is T-invariant, $V_i \perp V_j$ and $dim V_i = 1, 2$.

(2) Assume that $V = \mathbb{R}^3$, S its standard basis and T is the linear operator given by $T(x, y, z) = (-z, y, x)$. Verify that T is a unitary operator.

(3) Find a direct sum decomposition $V_1 \bigoplus V_2 \bigoplus \cdots \bigoplus V_r$ of V satisfying (1).

Exercise 4. (10-8-6-4)

Let V be a complex vector space of finite dimension and T a linear operator on V.

(1) Prove that T has a fan $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = V$.

Assume that $V = \mathbb{C}^3$ as a vector space over $\mathbb{C}$, S its standard basis and T the linear operator given by $T(x, y, z) = (x + iz, x + iy + z, ix + 3z)$.

(2) Find a fan of T.

(3) Find a basis B of V where the matrix $[T]_B$ representing T is upper triangular.

(4) Find an invertible matrix P such that $P^{-1}[T]_S P$ is upper triangular.

Exercise 5. (6-5-5-6)

Let $V = \mathbb{C}^3$ as a vector space over the complex field $\mathbb{C}$, and T the linear operator on V given by $T(x, y, z) = (x + y + z, y + z, z)$.

(1) Find the characteristic polynomial $f(X)$ of T.

(2) Verify that $f(T) = 0$.

(3) Show that T is invertible and find its inverse.

(4) Find the characteristic polynomial of T^{-1}.