

Major Exam I
Math 325 (Term 222, 2023)
KFUPM

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Throughout the whole exam the field acting on vector spaces is $\mathbf{K} \subset \mathbb{C}$.

Exercise 1. Say whether the following statements are true or false.

1. If $u \in \mathcal{L}(V)$ then $V = \ker u \oplus \text{Im}u$.
2. If u, v are linear maps such that $uv = 0$ then $u = 0$ or $v = 0$.
3. If $\sigma \in \text{Isom}(V)$ then $\text{rank}(\sigma v) = \text{rank}(v)$.
4. If $V + W = V + F$ then $W = F$.

Score partition: 1 pt each.

Exercise 2. Let $u \in \mathcal{L}(V)$ such that $\dim(V) = 3$, $u \neq 0$ and $u^2 = 0$.

1. Show that $\dim(\ker u) \neq 0, 3$.
2. Assume $\dim(\ker u) = 1$. Show there exists $y \in \text{Im}u - \{0\}$ such that $u(y) \neq 0$.
3. Conclude that $\dim(\ker u) = 2$.
4. Show that there exists $x \in V$ such that $(x, u(x))$ is linearly independent.

5. Deduce the existence of some appropriate basis \mathcal{B} of V in which

$$\text{Mat}_{\mathcal{B}}(u) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Score partition: 1.5 pt each.

Exercise 3. Let $A = (a_{i,j})$ be a non zero matrix of $M_n(\mathbf{K})$. We assume that

$$a_{i,j} = \alpha_i \beta_j$$

for some numbers $\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_n$.

1. What is the value of $\text{rank}(A)$?
2. Show there exist two matrices X :row and Y :column such that $A = XY$.
3. Show that $A^2 = \text{tr}(A)A$.
4. Deduce an easy expression for A^p for $p = 2, 3, \dots$
5. Assume that $\text{tr}(A)$ is not zero.
 - (a) Prove that $\mathbf{K}^n = \ker A \oplus \ker(A - \text{tr}(A)I_n)$.
 - (b) Conclude that A is similar to a diagonal matrix that you write down.

Score partition: 1.5 pt each.