Major Exam II
Math 325 (Term 222, 2023)
KFUPM

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Exercise 1. Consider $q(x, y, z) = xy + yz + z^2$ where $(x, y, z) \in \mathbb{R}^3$.

1. Say why q is a quadratic form.

2. Write down the canonical matrix of q.

3. Find the polar form Q of q.

4. Write q in terms of squares using Gauss's reduction method.

5. What is the signature of q.

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Exercise 2. Let $V = M_2(\mathbb{R})$ a consider the following bi-variate map

$$\varphi : \begin{array}{l} V \times V \to \mathbb{R} \\ (A, B) \mapsto \operatorname{tr}(A^T B) \end{array}.$$

1. Show that φ is a scalar product over V.

V is now a Euclidean space. Orthogonality is regarded with respect to φ.

2. Show that $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are orthogonal.

3. Let $W = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$.

(a) Find a basis for W.

(b) Show that $W^{\perp} = \text{span}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$.

(c) Give then an orthonormal basis for $W^{\perp}$.

4. Let $J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $p_{W^\perp}$ be the orthogonal projection onto $W^\perp$.

(a) Find $p_{W^\perp}(J)$.

(b) Deduce the distance between J and W.

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Exercise 3. Let $B = (e_1, ..., e_n)$ and $C = (f_1, ..., f_n)$ be two bases of some vector space V and let $B^* = (e_1^*, ..., e_n^*)$ and $C^* = (f_1^*, ..., f_n^*)$ their corresponding dual bases. Let P denote the transition matrix from B to C and similarly we define P^* to be the transition matrix from B^* to C^*. Note that a vector is a column when used in calculation. For every vector $u \in V$ and a basis L of V, the notation $[u]_L$ stands for the column $(u_1, ..., u_n)^T$ where the $u_i's$ are the coordinates of u in L.

1. What are the vectors $[e_j^*]_{B^*}$ and $[e_j]_B$?

2. Let $\varphi \in V^*$ and $x \in V$.

   (a) Show that $e_i^*(x) = [e_i^*]_{B^*}^T [x]_B$.

   (b) Conclude that $\varphi(x) = [\varphi]_{B^*}^T [x]_B$.

3. Show that $(P^*[e_i^*]_{B^*})(f_j) = \delta_{i,j}$.

4. Deduce that $[e_i^*]_{B^*}^T P^{*T} P [e_j]_B = \delta_{i,j}$.

5. Conclude the expression of P^*.

6. Let $V = \mathbb{R}^2$ equipped with its canonical basis $B = (e_1, e_2)$ and consider the dual basis C^* formed by $f_1^*(x, y) = x - 2y$ and $f_2^*(x, y) = x - y$.

   (a) Find $P_{C^* \to B^*}$.

   (b) Deduce $P_{B \to C}$.