Final Exam
Math 325 (Term 222, 2023)
KFUPM

by Dr. Maher Boudabra
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Full name:
Your ID:

- All scalar products are meant to be canonical unless otherwise mentioned. If needed, you may use the notations $< \cdot, \cdot >$ for the scalar product and $\|\cdot\|$ for its underlying norm.
- Recall that a sequence $(x_n)_n$ in $(V, < \cdot, \cdot >)$ converges to x means simply $\|x_n - x\| \underset{n \to +\infty}{\to} 0$. That is, feel free to use all standard limit theorems.

Exercise 1. Consider the real matrix

$$A = \begin{bmatrix} 0 & 1 \\ -a & 1+a \end{bmatrix}.$$

1. Compute the characteristic polynomial of A.
2. What is the spectrum of A?
3. Deduce that A is diagonalizable if and only if $a \neq 1$.
4. Find the eigenvectors of A .
5. Find the diagonal representation matrix of A together with the transition matrices.
6. Let (u_n) be the sequence defined recursively by

   $$u_{n+2} = (1+a)u_{n+1} - au_n$$

   with $u_0 = 0, u_1 = 1$ and let X_n be the vector $\begin{bmatrix} u_n \\ u_{n+1} \end{bmatrix}$.

(a) Show that $X_{n+1} = AX_n$.

(b) Express X_n in terms of A, X_0 and n.

(c) We suppose that $a \in (-1, 1)$. Find the limit of the sequence (u_n).

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Exercise 2. Consider the following real matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

1. Show that A is orthogonal.

2. Compute the characteristic polynomial of A.

3. Conclude that A acts as a rotation space.

4. Find a unit vector v such that $Av = v$.

5. Find $\cos(\theta)$ where θ is the angle of A.

6. Find θ.
   You can use without proof that $\sin(\theta)\det(v, x, f(x)) \geq 0$ for any vector non collinear to v.

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Exercise 3. Let u be an isometry of a Euclidean vector space V and set $v = u - id$.

1. Recall the definition of an isometry.

2. Show that the u^n is also an isometry for $n = 0, 1, 2, ...$

3. Show that $\ker v = \mathrm{Im} v^{\perp}$.

4. Deduce that $\ker v^{\perp} = \mathrm{Im} v$.

5. For every integer n define the linear map u_n by

$$u_n = \frac{1}{n}\sum_{k=0}^{n-1} u^k.$$

   (a) Let $x \in \ker v$. Show that $u_n(x) = x$.

   (b) Let $x \in \ker v^{\perp}$. Show that $u_n(x) = \frac{1}{n}(u^n(z) - z)$ for some vector $z \in V$.

6. Show that $u_n(x) \underset{n\to+\infty}{\rightarrow} 0$ for $x \in \ker v^{\perp}$.

7. Identify the limit of $u_n(x)$ for any vector $x \in V$.
   Remember that $V = \ker v \oplus \ker v^{\perp}$.

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Exercise 4. Let v be a diagonalizable endomorphism of K^n whose characteristic polynomial is

$$P_v(x) = \Pi_{i=1}^{r}(x - \lambda_i)^{n_i}$$

with the n_i's are positive. We define the commutant of v by

$$C(v) = \{u \in \mathcal{L}(K^n) \mid uv = vu\}$$

and we set $V_i := \ker(v - \lambda_i id)$, $v_i := v_{|V_i}$. Recall that

$$K^n = V_1 \oplus V_2 \oplus \cdots \oplus V_r$$

1. What is $C(id)$?
2. Find $\dim C(id)$.
3. Show that $K[v] \subset C(v)$ where $K[v] := \{f(v) \mid f \in K[x]\}$.
4. What is the minimal polynomial of v.
5. Find $\dim V_i$.
6. What is $v_i(x)$ for $x \in V_i$?
7. Let u be an element of $C(v)$. Show that V_i is u-stable.
8. Let u be an endomorphism of K^n. Show that if all the V_i's are u-stable then u belongs to $C(v)$.
9. Conclude that
$$\begin{aligned} \Phi : C(v) &\longrightarrow \mathcal{L}(V_1) \times \mathcal{L}(V_2) \times \cdots \times \mathcal{L}(V_r) \\ u &\longmapsto (u_{|V_1}, u_{|V_2}, ..., u_{|V_r}) \end{aligned}$$
is an isomorphism.
10. Show that $\dim C(v) = n_1^2 + n_2^2 + \cdots + n_r^2$.
11. Show that $\dim C(v) \geq n$.
12. Find $\dim C(v)$ where v is represented by the matrix
$$A = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}.$$