King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 333 Major Exam 1
The Second Semester of 2012-2022 (211)
Time Allowed: 120 Minutes

Solution

Q:1 (4+4 points) Consider the vector function $\vec{r}(t) = t^3\,\hat{i} + \sqrt{3}\,t^2\,\hat{j} + 2t\,\hat{k}$.

(a) Find length of the curve traced by $\vec{r}(t)$ for $0 \le t \le 1$.

(b) Find parametric equations of the tangent line to the curve traced by $\vec{r}(t)$ at $t = 1$.

Sol: (a) $\vec{r}\,'(t) = 3t^2\,\hat{i} + 2\sqrt{3}\,t\,\hat{j} + 2\hat{k}$

$$L = \int_0^1 \|\vec{r}\,'(t)\|\,dt = \int_0^1 \sqrt{9t^4 + 12t^2 + 4}\,dt \quad ②$$

$$= \int_0^1 \sqrt{(3t^2+2)^2}\,dt = \int_0^1 (3t^2+2)\,dt$$

$$= t^3 + 2t \Big|_0^1 = 1 + 2 = 3 \quad ②$$

(b) $\vec{r}(1) = 1\hat{i} + \sqrt{3}\,\hat{j} + 2\hat{k}$ $\qquad P(1, \sqrt{3}, 2)$ ①

$\vec{r}\,'(1) = 3\hat{i} + 2\sqrt{3}\,\hat{j} + 2\hat{k}$ ①

The parametric equations of the tangent line are

$$x(t) = 1 + 3t, \quad y = \sqrt{3} + 2\sqrt{3}\,t, \quad z = 2 + 2t \quad ②$$

Q:2 (4+4 points) Consider the function $F(x, y, z) = \sqrt{x^2y + 2y^2z}$.

(a) Compute the directional derivative of F at $(-2, 2, 1)$ in the direction of the negative z-axis.

(b) Find the direction along which F **decreases** most rapidly at the point $(2, 1, 0)$. Also find the minimum value of the rate of change at this point.

Sol: (a) $$\nabla F = \frac{\cancel{2}xy}{\cancel{2}\sqrt{x^2y+2y^2z}}\,\hat{i} + \frac{x^2+4yz}{2\sqrt{x^2y+2y^2z}}\,\hat{j} + \frac{\cancel{2}y^2}{\cancel{2}\sqrt{x^2y+2y^2z}}\,\hat{k}$$

$$\nabla F(-2,2,1) = \frac{-4}{\sqrt{8+8}}\,\hat{i} + \frac{4+8}{2\sqrt{8+8}}\,\hat{j} + \frac{4}{\sqrt{8+8}}\,\hat{k}$$

$$= -\hat{i} + \frac{3}{2}\hat{j} + \hat{k} \qquad 3$$

$$\hat{u} = -\hat{k}$$

$$D_{\hat{u}} F(-2,2,1) = (1)(-1) = -1 \qquad 1$$

(b) At $(2,1,0)$, F decreases most rapidly in the direction

$$-\nabla F(2,1,0) = -\frac{2}{\sqrt{4}}\,\hat{i} - \frac{4}{2\sqrt{4}}\,\hat{j} - \frac{1}{\sqrt{4}}\,\hat{k}$$

$$= -\hat{i} - \hat{j} - \frac{1}{2}\hat{k} \qquad 2$$

The minimum rate of change is

$$-\|\nabla F(2,1,0)\| = -\sqrt{1+1+\frac{1}{4}} = -\frac{3}{2} \qquad 2$$

Q:3 (4+3 points) Consider the vector field $\vec{F}(x,y,z) = x^2y\ \hat{\imath} + xy^2\ \hat{\jmath} + 2xyz\ \hat{k}$. Find the following:

(a) $\|\nabla \times \vec{F}\|$ at the point $(1,1,1)$.

(b) $\nabla(\nabla \cdot \vec{F})$.

Sol: (a) $\text{Curl}\ \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xy^2 & 2xyz \end{vmatrix}$ (2)

$= (2xz - 0)\hat{i} - (2yz - 0)\hat{j} + (y^2 - x^2)\hat{k}$

$= 2xz\,\hat{i} - 2yz\,\hat{j} + (y^2 - x^2)\hat{k}$ (1)

$\nabla \times \vec{F}\,(1,1,1) = 2\hat{i} - 2\hat{j} + 0\hat{k}$

$\|\nabla \times \vec{F}\| = \sqrt{4+4} = 2\sqrt{2}$ (1)

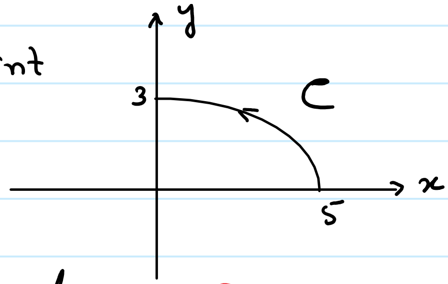
(b) $\nabla \cdot \vec{F} = 2xy + 2xy + 2xy = 6xy$ (2)

$\nabla(\nabla \cdot \vec{F}) = 6y\,\hat{i} + 6x\,\hat{j} + 0\hat{k}$ (1)

Q:4 (10 points) Find the work done by the force $\vec{F}(x,y,z) = (9-y^2)\,\hat{\imath} + xy\,\hat{\jmath}$ acting along the curve C given by $\frac{x^2}{25} + \frac{y^2}{9} = 1$ from the point $(5,0)$ to the point $(0,3)$.

Sol: $x(t) = 5\cos t\,,\ y(t) = 3\sin t$

$0 \le t \le \frac{\pi}{2}$ 2



$$W = \int_C \vec{F}\cdot d\vec{r} = \int_C (9-y^2)\,dx + xy\,dy$$ 2

$$= \int_0^{\frac{\pi}{2}} (9 - 9\sin^2 t)(-5\sin t)\,dt + 5\cos t \cdot 3\sin t \cdot 3\cos t\,dt$$

$$= \int_0^{\frac{\pi}{2}} (-45\cos^2 t\,\sin t + 45\cos^2 t\,\sin t)\,dt$$ 3

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$$= 0$$

Q:5 (3+6+3 points) Let $\vec{F}(x,y,z) = (e^x \sin y - yz)\,\hat{\imath} + (e^x \cos y - xz)\,\hat{\jmath} + (z - xy)\,\hat{k}$

(a) Show that $\vec{F}$ is conservative.

(b) Find the potential function $\phi(x,y,z)$ such that $\nabla\phi = \vec{F}(x,y,z)$.

(c) Compute $\displaystyle\int_{(0,\frac{\pi}{6},2)}^{(0,\frac{\pi}{2},4)} \vec{F}\cdot d\vec{r}$ using the function $\phi(x,y,z)$.

Sol: (a) $P(x,y,z) = e^x \sin y - yz$, $Q(x,y,z) = e^x \cos y - xz$

$R(x,y,z) = z - xy$

$$\frac{\partial Q}{\partial x} = e^x \cos y - z = \frac{\partial P}{\partial y}, \quad \frac{\partial R}{\partial y} = -x = \frac{\partial Q}{\partial z}$$

$$\frac{\partial R}{\partial x} = -y = \frac{\partial P}{\partial z}$$

$\Rightarrow \vec{F}$ is conservative. 3

(b) Find $\phi(x,y,z)$ such that $\vec{F} = \nabla\phi$

$$\frac{\partial \phi}{\partial x} = P = e^x \sin y - yz, \quad \frac{\partial \phi}{\partial y} = Q = e^x \cos y - xz$$

$$\frac{\partial \phi}{\partial z} = z - xy$$

3

$$\phi(x,y,z) = e^x \sin y - xyz + g(y,z)$$

$$\frac{\partial \phi}{\partial y} = e^x \cos y - xz + \frac{\partial g}{\partial y} = Q = e^x \cos y - xz$$

$$\Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g = h(z)$$

$$\Rightarrow \quad \frac{\partial g}{\partial y} = 0 \quad \Rightarrow g = h(z)$$

So $\quad \phi(x,y,z) = e^x \sin y - xyz + h(z)$

$$\frac{\partial \phi}{\partial z} = -xy + h'(z) = R = z - xy$$

$$\Rightarrow \quad h'(z) = z \quad \Rightarrow h(z) = \frac{z^2}{2}$$

$$\phi(x,y,z) = e^x \sin y - xyz + \frac{1}{2} z^2 \qquad 3$$

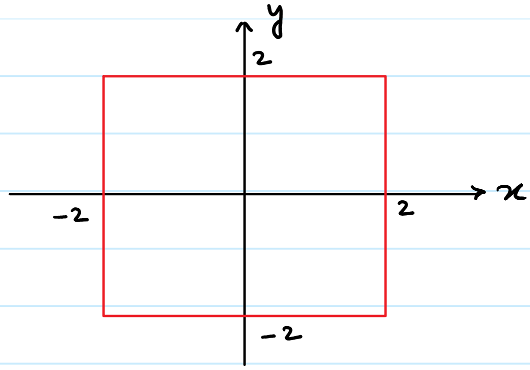
(c) $$\int_{(0,\frac{\pi}{6},2)}^{(0,\frac{\pi}{2},4)} \vec{F} \cdot d\vec{r} = \phi(0,\tfrac{\pi}{2},4) - \phi(0,\tfrac{\pi}{6},2)$$

$$= (1 - 0 + 8) - (\tfrac{1}{2} - 0 + 2)$$

$$= 9 - \frac{5}{2} = \frac{18-5}{2} = \frac{13}{2} \qquad 3$$

Q:6 (10 points) Evaluate the integral $\displaystyle\oint_C \frac{x^2y\,dx - x^3\,dy}{(x^2+y^2)^2}$, where C is the positively oriented boundary of the region $-2 \le x \le 2$ and $-2 \le y \le 2$.

Sol: We cannot apply Green's theorem.



$$P = \frac{x^2y}{(x^2+y^2)^2}, \quad Q = \frac{-x^3}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{-3x^2(x^2+y^2)^2 + x^3\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}$$

$$= \frac{-3x^2(x^2+y^2) + 4x^4}{(x^2+y^2)^3} = \frac{x^4 - 3x^2y^2}{(x^2+y^2)^3} \qquad 2$$

$$\frac{\partial P}{\partial y} = \frac{x^2(x^2+y^2)^2 - x^2y\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}$$

$$= \frac{x^2(x^2+y^2) - 4x^2y^2}{(x^2+y^2)^3} = \frac{x^4 - 3x^2y^2}{(x^2+y^2)^3} \qquad 2$$

Integral independent of path

Consider C_1 : $x = \cos t$, $y = \sin t$ $\quad 0 \le t \le 2\pi$ $\quad 2$

$$\oint_C P\,dx + Q\,dy = \oint_{C_1} P\,dx + Q\,dy$$

$$= \int_0^{2\pi} \cos^2 t \sin t(-\sin t)\,dt - \cos^3 t \cos t\,dt = -\int_0^{2\pi} \cos^2 t\,dt$$

$$= -\frac{1}{2}\int_0^{2\pi} (1+\cos 2t)\,dt = -\frac{1}{2}\left[t + \frac{\sin 2t}{2}\right]\Big|_0^{2\pi} = -\pi \qquad 4$$

Q:7 (10 points) Evaluate the integral $\oint \vec{F}\cdot d\vec{r}$ by using the Stokes' theorem,

Q:7 (10 points) Evaluate the integral $\oint_C \vec{F} \cdot d\vec{r}$ by using the Stokes' theorem,

where $\vec{F} = 2y^3\,\hat{i} - 2x^3\,\hat{j} + \tan^{-1}(z)\,\hat{k}$ and C is the trace of cylinder $x^2 + y^2 = 1$ in the plane $x + y + z = 1$.

Sol:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y^3 & -2x^3 & \tan^{-1} z \end{vmatrix} = 0\hat{i} - 0\hat{j} + (-6x^2 - 6y^2)\hat{k} \quad 2$$

$$\hat{n} = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k}) \quad 1, \qquad z = 1 - x - y$$

$$ds = \sqrt{1+1+1}\, dA \quad 1$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds \quad 2$$

$$= \iint_A \frac{-6(x^2+y^2)}{\sqrt{3}} \cdot \sqrt{3}\, dA \quad 1$$

$$= -6 \int_0^{2\pi} \int_0^1 r^2 \cdot r\, dr\, d\theta \quad 1$$

$$= -\frac{6}{4} r^4 \Big|_0^1 \cdot \theta \Big|_0^{2\pi} = -\frac{3}{2} \cdot 2\pi = -3\pi$$

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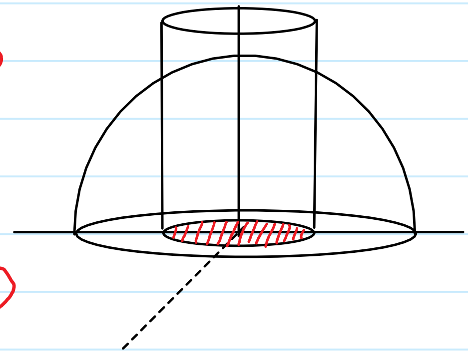
Q:8 (10 points) Use divergence theorem to evaluate $\iint\limits_S (\vec{F}.\hat{n})\, dS$ where

$$\vec{F} = z^3\cos^2(y)\,\hat{i} + \sin^3(x)z^2\,\hat{j} + z^3\,\hat{k}$$

and D is the region bounded within by $z = \sqrt{9 - x^2 - y^2}$, $x^2 + y^2 = 4$ and $z = 0$.

Sol: $\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = 0 + 0 + 3z^2 = 3z^2$ ②

$\iint\limits_S \vec{F}\cdot\hat{n}\,ds = \iiint\limits_D \nabla\cdot\vec{F}\,dv$ ②



$$= \int_0^{2\pi}\int_0^{2}\int_0^{\sqrt{9-r^2}} 3z^2\, r\,dz\,dr\,d\theta$$ ③

$$= 2\pi \int_0^2 (9-r^2)^{\frac{3}{2}}\, r\,dr$$

$$= \frac{2\pi}{-2}\,\frac{2}{5}(9-r^2)^{\frac{5}{2}}\Big|_0^2 = -\frac{2\pi}{5}\left(5^{\frac{5}{2}} - 3^5\right)$$ ③

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