King Fahd University of Petroleum and Minerals
Department of Mathematics

# MATH 333 - Exam 1 - Term 212

Duration: 120 minutes

Name: Answer Key ID Number:______

Section Number:______ Serial Number:______

Class Time:______ Instructor's Name:______

**Instructions:**

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 9 pages of problems. (Total of 8 Problems)

| Question # | Points | Maximum Points |
|---|---|---|
| 1 | | 8 |
| 2 | | 7 |
| 3 | | 10 |
| 4 | | 11 |
| 5 | | 9 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| Total | | 75 |

1. **[4+4 points]** Consider the vector function $\vec{r}(t) = \cos(3t)\,\vec{i} + \sin(3t)\,\vec{j} + 2t^{\frac{3}{2}}\,\vec{k}$.

(a) Find the length of the curve traced by $\vec{r}(t)$ for $0 \le t \le 3$.

(b) Find parametric equations of the tangent line to the curve traced by $\vec{r}(t)$ at $t = \pi$.

(a) $\vec{r}\,'(t) = -3\sin(3t)\,\vec{i} + 3\cos(3t)\,\vec{j} + 3t^{\frac{1}{2}}\,\vec{k}$ 1

$L = \int_0^3 \|\vec{r}\,'(t)\|\,dt = \int_0^3 \sqrt{9\sin^2(3t) + 9\cos^2(3t) + 9t}\;dt$ 1

$= 3\int_0^3 \sqrt{1+t}\;dt$

$= 2\left[(1+t)^{\frac{3}{2}}\right]_0^3$ 1

$= 2\left[4^{\frac{3}{2}} - 1\right]$

$= 14.$ 1

(b) $\vec{r}(\pi) = -\vec{i} + 0\,\vec{j} + 2\pi^{\frac{3}{2}}\,\vec{k}$ 1, $\vec{r}\,'(\pi) = 0\,\vec{i} - 3\,\vec{j} + 3\pi^{\frac{1}{2}}\,\vec{k}$ 1

The parametric equations of the tangent Line are

$x = -1 + 0t$

$y = 0 - 3t$

$z = 2\pi^{\frac{3}{2}} + 3\pi^{\frac{1}{2}}t$

OR

$x = -1$

$y = -3t$ 2

$z = 2\pi^{\frac{3}{2}} + 3\pi^{\frac{1}{2}}t.$ ⋇

2. **[7 points]** The temperature in a certain 3 dimensional solid is given by

$$T(x,y,z) = 2z^3 - 3(x^2+y^2)z + \tan^{-1}(xz)$$

If a mosquito is located at the point $(-1,1,1)$, in which direction should it fly to cool off as rapidly as possible?

$$\nabla T(x,y,z) = \left(-6xz + \frac{z}{1+x^2z^2}\right)\vec{i} + (-6yz)\vec{j} + \left(6z^2 - 3x^2 - 3y^2 + \frac{x}{1+x^2z^2}\right)\vec{k}$$

3

$$\nabla T(-1,1,1) = \left(6 + \frac{1}{1+1}\right)\vec{i} - 6\vec{j} + \left(6 - 3 - 3 - \frac{1}{1+1}\right)\vec{k}$$

$$= \frac{13}{2}\vec{i} - 6\vec{j} - \frac{1}{2}\vec{k}$$

2

The mosquito should fly in the direction of $-\nabla T(-1,1,1)$, that is, in the direction of

$\left\langle \frac{-13}{2}, +6, +\frac{1}{2} \right\rangle$. *

2

3. **[10 points]** Find the work done by the force

$$\vec{F}(x,y,z) = x\,\vec{i} + z^2\,\vec{j} - y\,\vec{k}$$

on a particle that moves along the segment line from the point (2, 3, 3) to (6, 4, 5).

$$\text{Work} = \int_C \vec{F}\cdot d\vec{r} = \int_C x\,dx + z^2\,dy - y\,dz$$

The parametric equations of the segment Line are

$$x = 2+4t \quad , \quad y = 3+t \quad , \quad z = 3+2t \quad ; 0 \le t \le 1$$

$$\Rightarrow dx = 4\,dt \; ① \quad , \quad dy = dt \; ① \quad , \quad dz = 2\,dt \; ①$$

② 

$$\text{Work} = \int_0^1 (2+4t)\cdot 4\,dt + (3+2t)^2\,dt - (3+t)\cdot 2\,dt$$

$$= \left[ 8t + 8t^2 + \frac{(3+2t)^3}{6} - 6t - t^2 \right]_0^1 \quad ③$$

$$= 8 + 8 + \frac{125}{6} - 6 - 1 - \frac{9}{2}$$

$$= \frac{9}{2} + \frac{125}{6}$$

$$= \frac{152}{6} \quad ②$$

$$= \frac{76}{2}. \quad \#$$

4. **[2+6+3 points]** Consider the vector field

$$\vec{F}(x,y,z) = (2-e^z)\,\vec{i} + (2y-1)\,\vec{j} + (2-xe^z)\,\vec{k}.$$

(a) Show that $\vec{F}$ is conservative.

(b) Find a potential function $\phi$ for $\vec{F}$.

(c) Use part (b) to evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$ where $C$ is any path that starts from the point $(-1,1,-1)$ and ends at $(2,4,8)$.

(a)

$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2-e^z & 2y-1 & 2-xe^z \end{vmatrix}$$

$$= (0-0)\vec{i} - (-e^z + e^z)\vec{j} + (0-0)\vec{k} \quad 2$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}.$$

$$=$$

Hence $\vec{F}$ is conservative.

(b) There exists a function $\phi(x,y,z)$ such that

$$\frac{\partial \phi}{\partial x} = 2 - e^z \quad \cdots (1), \qquad \frac{\partial \phi}{\partial y} = 2y-1 \quad \cdots (2), \qquad \frac{\partial \phi}{\partial z} = 2 - xe^z \quad \cdots (3)$$

Integrate the both sides of equation (1) w.r.t. $x$:

$$\phi(x,y,z) = 2x - xe^z + g(y,z) \quad \cdots (4) \quad 3$$

Differentiate the both sides of equation (4) w.r.t. $y$:

$$\frac{\partial \phi}{\partial y} = \frac{\partial g}{\partial y}(y,z) \overset{(2)}{=} 2y-1 \Rightarrow g(y,z) = y^2 - y + h(z)$$

2

substitute $g$ in equation ④

$$\phi(x,y,z) = 2x - xe^z + y^2 - y + h(z) \quad \cdots ⑤$$

Differentiate the both sides of equation ⑤ w.r.t $z$:

$$\frac{\partial\phi}{\partial z} = \cancel{-xe^z} + h'(z) \stackrel{③}{=} 2 \cancel{-xe^z}$$

$$\Rightarrow h'(z) = 2 \Rightarrow h(z) = 2z.$$

substituting back to equation ⑤ gives

$$\phi(x,y,z) = 2x - xe^z + y^2 - y + 2z. \quad 1$$

ⓒ $$\int_C \vec{F}\cdot d\vec{r} = \phi(2,4,8) - \phi(-1,1,-1) \quad 1.5$$

$$= (4 - 2e^8 + 16 - 4 + 16) - (-2 + e^{-1} + 1 - 1 - 2)$$

$$= 32 - 2e^8 + 4 - e^{-1}$$

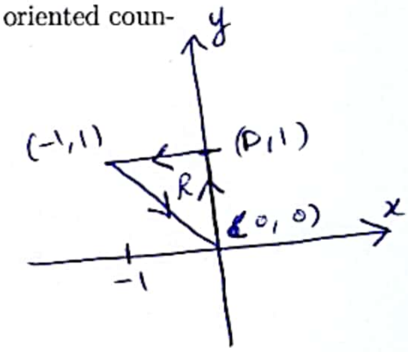
$$= 36 - 2e^8 - e^{-1}. \quad 1.5$$

#

5. **[9 points]** Use Green's theorem to evaluate the integral

$$\oint_C e^{x^2}\,dx + 2\tan^{-1} x\,dy$$

where $C$ is the triangle with vertices $(0,0)$, $(0,1)$ and $(-1,1)$ and oriented counterclockwise.




$P(x,y) = e^{x^2}$ , $Q(x,y) = 2\tan^{-1} x$

$\dfrac{\partial P}{\partial y} = 0$ , $\dfrac{\partial Q}{\partial x} = \dfrac{2}{1+x^2}$ 2

According to Green's theorem; $\oint_C P\,dx + Q\,dy = \iint_R \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right) dA$

$$\oint_C e^{x^2}dx + 2\tan^{-1}x\,dy = \iint_R \frac{2}{1+x^2}\,dA \quad 1$$

$$= \int_{-1}^{0}\int_{-x}^{1} \frac{2}{1+x^2}\,dy\,dx \quad 2$$

$$= 2\int_{-1}^{0} \left[\frac{y}{1+x^2}\right]_{y=-x}^{y=1} dx$$

$$= 2\int_{-1}^{0} \frac{1}{1+x^2} + \frac{x}{1+x^2}\,dx$$

$$= 2\left[\tan^{-1}x + \frac{1}{2}\ln(1+x^2)\right]_{-1}^{0} \quad 2$$

$$= 2\left[0+0 - \left(-\frac{\pi}{4} + \frac{1}{2}\ln 2\right)\right]$$

$$= \frac{\pi}{2} - \ln 2. \quad 2$$

6. **[10 points]** Evaluate the surface integral

$$\iint_S 2z\, dS,$$

where $S$ is that portion of the sphere $x^2+y^2+z^2=25$ that is above the region in the first quadrant bounded by $x=0$, $y=0$ and $x^2+y^2=16$.

$x^2+y^2+z^2=25 \Rightarrow z=\sqrt{25-x^2-y^2}$

so $f(x,y)=\sqrt{25-x^2-y^2}$ 2

$f_x=\dfrac{-\cancel{2}x}{\cancel{2}\sqrt{25-x^2-y^2}}$ , $f_y=\dfrac{-\cancel{2}y}{\cancel{2}\sqrt{25-x^2-y^2}}$ 2

$dS=\sqrt{1+f_x^2+f_y^2}\, dA=\sqrt{1+\dfrac{x^2}{25-x^2-y^2}+\dfrac{y^2}{25-x^2-y^2}}\, dA$

1

$=\dfrac{5}{\sqrt{25-x^2-y^2}}\, dA$ 1

$\iint_S 2z\, dS=\iint_R 2\cdot\sqrt{25-x^2-y^2}\cdot\dfrac{5}{\sqrt{25-x^2-y^2}}\, dA$ 2

$=10\iint_R dA$

$=10\,\text{Area}(R)$

2

$=10\cdot\dfrac{16\pi}{4}=40\pi.$ #

7. **[10 points]** Use Stokes' theorem to evaluate the integral $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS$,

where
$$\vec{F}(x,y,z) = 2xy\,\vec{i} + y^2 z\,\vec{j} + y\tan^{-1}(x^2)\vec{k}$$

and $S$ is the portion of the paraboloid $z = \dfrac{x^2}{4} + \dfrac{y^2}{9}$, for $0 \le z \le 1$ **in the first octant.**

According to Stokes' theorem

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, ds = \oint_C \vec{F} \cdot d\vec{r} \quad 1$$

where $C: \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ ; $z = 1$.

Let $x = 2\cos t$ and $y = 3\sin t$ , $z = 1$ 2

$dx = -2\sin t\, dt$ , $dy = 3\cos t\, dt$ , $dz = 0$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C 2xy\, dx + y^2 z\, dy + y\tan^{-1}(x^2)\, dz \quad 1$$

$$= \int_0^{\pi/2} 2 \cdot 6\cos t \cdot \sin t \cdot (-2\sin t)\, dt \quad 3$$
$$+ \int_0^{\pi/2} 9\sin^2 t \cdot 3\cos t\, dt$$

$$= \int_0^{\pi/2} 3\sin^2 t \cos t\, dt$$

$$= \sin^3 t \Big]_0^{\pi/2} = 1. \quad 2$$

#

Scanned with CamScanner

8. **[10 points]** Use divergence theorem to find the outward flux $\iint_S (\vec{F}\cdot\vec{n})\,dS$ of the vector field

$$\vec{F}(x,y,z) = (2xy+\sin^3 z)\,\vec{i} + (x^2+y^2)\,\vec{j} + (x^3 y + \cos x)\,\vec{k}$$

through the region bounded by the surfaces $x = y^2$, $z = 4 - x$ and $z = 0$.

$\mathrm{div}(\vec{F}) = \nabla\cdot\vec{F} = 2y + 2y + 0 = 4y.$ 2

According to Divergence theorem:

$$\iint_S (\vec{F}\cdot\vec{n})\,ds = \iiint_D \nabla\cdot\vec{F}\,dV \quad 2$$

$$= \iiint_D 4y\,dV$$

$$= \int_0^2\int_{y^2}^4\int_0^{4-x} 4y\,dz\,dx\,dy \quad 3$$

$$= \int_0^2\int_{y^2}^4 4y(4-x)\,dx\,dy$$

$$= \int_0^2 4y\left(4x - \frac{x^2}{2}\right)\Big]_{x=y^2}^{x=4} dy$$

$$= \int_0^2 4y\left(16 - 8 - 4y^2 + \frac{y^4}{2}\right) dy$$

$$= \int_0^2 (32y - 16y^3 + 2y^5)\,dy$$

$$= 16y^2 - 4y^4 + \frac{1}{3}y^6 \Big]_0^2$$

$$= 64 - 64 + \frac{64}{3}$$

3

$$= \frac{64}{3}. \quad \#$$