King Fahd University of Petroleum and Minerals
Department of Mathematics

MATH 333 - Exam 2 - Term 212

Duration: 120 minutes

Name: Key ID Number: ____________

Section Number: ____________ Serial Number: ____________

Class Time: ____________ Instructor's Name: ____________

Instructions:

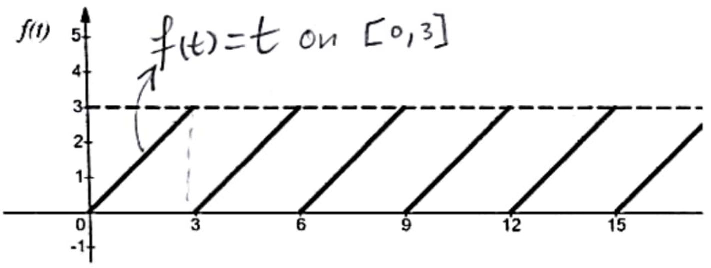
1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 9 pages of problems and the formula sheet. (Total of 7 Problems)

Question #	Points	Maximum Points
1		12
2		8
3		9
4		16
5		11
6		9
7		10
Total		75

1. **[6+6 points]** Find the Laplace transform of the following

(a) $f(t) = (t + \cos t)^2$.

(b) the function $f(t)$ defined on the interval $[0, \infty)$ and whose graph is given below



ⓐ $f(t) = t^2 + 2t\cos t + \cos^2 t = t^2 + 2t\cos t + \frac{1}{2} + \frac{1}{2}\cos(2t)$ (1)

$$\mathcal{L}\{2t\cos t\} = -2\frac{d}{ds}\mathcal{L}\{\cos t\} = -2\frac{d}{ds}\left(\frac{s}{s^2+1}\right) \quad (1)$$

$$= -2\,\frac{s^2+1-2s^2}{(s^2+1)^2} \quad (1)$$

$$= \frac{2s^2-2}{(s^2+1)^2}$$

So

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} + \mathcal{L}\{2t\cos t\} + \mathcal{L}\{\tfrac{1}{2}\} + \mathcal{L}\{\tfrac{1}{2}\cos 2t\}$$

$$= \frac{2}{s^3} \ (1) + \frac{2s^2-2}{(s^2+1)^2} + \frac{1}{2s} \ (1) + \frac{1}{2}\,\frac{s}{s^2+4} \ (1). \qquad \#$$

ⓑ The function f is periodic with period $T = 3$ (1) where $f(t) = t$ on $[0,3]$.

$$\begin{array}{c|c} t & e^{-st} \\ 1 & -\frac{1}{s}e^{-st} \\ 0 & \frac{1}{s^2}e^{-st} \end{array}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-sT} f(t)\,dt$$

$$= \frac{1}{1-e^{-3s}}\int_0^3 t\,e^{-st}\,dt \quad (1)$$

$$= \frac{1}{1-e^{-3s}}\left[-\frac{t}{s}e^{-st} \ (1) - \frac{1}{s^2}e^{-st} \ (1)\right]_0^3.$$

$$= \frac{1}{1-e^{-3s}}\left[-\frac{3}{s}e^{-3s} - \frac{1}{s^2}e^{-3s} + \frac{1}{s^2}\right]. \quad (2) \qquad \#$$

2. **[8 points]** Solve the initial value problem using Laplace transform

$$y'' + y = \delta(t - 2\pi), \; y(0) = 1, \; y'(0) = 0.$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\delta(t-2\pi)\}$$

$$\Rightarrow s^2 Y - s y(0) - y'(0) + Y = e^{-2\pi s} \quad 2$$

$$\Rightarrow (s^2+1) Y = s + e^{-2\pi s}$$

$$\Rightarrow Y = \frac{s}{s^2+1} + \frac{1}{s^2+1} e^{-2\pi s} \quad 2$$

We take the inverse Laplace transform:

$$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} e^{-2\pi s}\right\}$$

$$\Rightarrow y = \cos t + \mathcal{U}(t-2\pi) \, \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\Big|_{t \to t-2\pi}$$

1 1 1

$$= \cos t + \sin(t-2\pi) \, \mathcal{U}(t-2\pi)$$

$$= \cos t + \sin t \, \mathcal{U}(t-2\pi). \quad \#$$

1

3. **[9 points]** Use Laplace transform to find the solution of the integral equation

$$f(t) + \int_0^t f(t-\tau)d\tau = g(t), \text{ where } g(t) = \begin{cases} 1, & 0 \le t < 2 \\ 2, & 2 \le t \end{cases}.$$

First, notice that

① $\int_0^t f(t-\tau)\,d\tau = f(t) * 1$

② $g(t) = 1 - \mathcal{U}(t-2) + 2\,\mathcal{U}(t-2) = 1 + \mathcal{U}(t-2)$ ①

Now we apply the Laplace transform on the integral equation

$$\mathcal{L}\{f(t)\} + \mathcal{L}\{1 * f(t)\} = \mathcal{L}\{1 + \mathcal{U}(t-2)\}$$

$$\Rightarrow \quad F(s) + \frac{F(s)}{s} = \frac{1}{s} + \frac{1}{s}e^{-2s} \quad ②$$

$$\Rightarrow \quad F(s) = \frac{1}{s+1} + \frac{1}{s+1}e^{-2s} \quad ②$$

We take the inverse Laplace transform:

$$f(t) = e^{-t} + \mathcal{U}(t-2)\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}\Big|_{t \to t-2}$$

$$= e^{-t} + e^{-(t-2)}\,\mathcal{U}(t-2). \quad \#$$

① ① ① ①

4. **[12+4 points]** (a) Find the Fourier series of the function

$$f(x) = \begin{cases} \dfrac{2x}{\pi}, & 0 \le x < \pi, \\ 2, & \pi \le x \le 2\pi \end{cases}.$$

(b) Use the Fourier series in part (a) to show that $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \ldots = \frac{\pi^2}{8}$.

(a) $$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} \frac{2x}{\pi}\,dx + \int_{\pi}^{2\pi} 2\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{x^2}{\pi}\right]_0^{\pi} + 2\pi\right] \quad 1$$

$$= \frac{1}{\pi}\left[\pi + 2\pi\right] = 3. \quad 1$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi}{L}x\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \frac{2x}{\pi}\cos nx\,dx + \int_{\pi}^{2\pi} 2\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[\underbrace{\frac{2x}{n\pi}\sin nx + \frac{2}{\pi n^2}\cos nx}_{2}\right]_0^{\pi} + \left[\frac{2}{n}\sin nx\right]_{\pi}^{2\pi}\right] \quad 1$$

$$= \frac{1}{\pi}\left[\frac{2(-1)^n}{\pi n^2} - \frac{2}{\pi n^2}\right]$$

$$= \frac{2}{\pi^2 n^2}\left((-1)^n - 1\right) \quad 1$$

$\frac{2x}{\pi}$		$\cos nx$
	$+$	
$\frac{2}{\pi}$		$\frac{1}{n}\sin nx$
	$-$	
0		$\frac{-1}{n^2}\cos nx$

$$b_n = \frac{2}{L}\int_0^L f(x)\ \sin\frac{2n\pi}{L}x\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi}\frac{2x}{\pi}\sin nx\,dx + \int_{\pi}^{2\pi} 2\sin nx\,dx\right]$$

$\frac{2x}{\pi}$	$\sin nx$
	$+$
$\frac{2}{\pi}$	$-\frac{1}{n}\cos nx$
	$-$
0	$-\frac{1}{n^2}\sin nx$

$$= \frac{1}{\pi}\left[\left[-\frac{2x}{\pi n}\cos nx + \frac{2}{\pi n^2}\sin nx\right]_0^{\pi} - \left[\frac{2}{n}\cos nx\right]_{\pi}^{2\pi}\right]$$ ② ①

$$= \frac{1}{\pi}\left[-\frac{2\pi}{\pi n}(-1)^n - \frac{2}{n} + \frac{2}{n}(-1)^n\right]$$

$$= -\frac{2}{\pi n}$$ ①

The Fourier series of f is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi}{L}x + b_n\sin\frac{2n\pi}{L}x\right)$$

$$= \frac{3}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{\pi^2 n^2}\left((-1)^n-1\right)\cos nx - \frac{2}{n\pi}\sin nx\right].$$ ② #

ⓑ At $x=0$ the series converges to $\frac{f(0^+)+f(0^-)}{2} = \frac{0+2}{2} = 1$. ①

So substituting $x=0$ in the above series gives

$$\frac{3}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi^2 n^2}\left((-1)^n-1\right) = 1$$

②

$$\Rightarrow \frac{3}{2} + \frac{2}{\pi^2}\left[-\frac{2}{1} - \frac{2}{3^2} - \frac{2}{5^2} - \frac{2}{7^2} - \cdots\right] = 1$$

$$\Rightarrow -\frac{4}{\pi^2}\left[1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots\right] = -\frac{1}{2}$$

①

$$\Rightarrow 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots = \frac{\pi^2}{8}.$$ #

5. **[11 points]** Find the eigenvalues and corresponding eigenfunctions of the following boundary value problem

$$x^2y'' + xy' + \lambda y = 0, \quad y'(1) = 0, \; y'(2) = 0.$$

The given d.e. is Cauchy-Euler equation and the corresponding auxiliary equation is

$r(r-1) + r + \lambda = 0 \Rightarrow r^2 + \lambda = 0$. 1

Case 1: If $\lambda = 0$ then we have $r^2 = 0 \Rightarrow r = 0, 0$ and hence

$y(x) = c_1 + c_2 \ln x$ 1 and $y' = \frac{c_2}{x}$

$y'(1) = 0 \Rightarrow \boxed{c_2 = 0}$ and $y'(2) = 0 \Rightarrow \boxed{c_2 = 0}$

so $y(x) = c_1$ is a nontrivial solution.

Hence $\lambda = 0$ is an eigenvalue corresponding to the eigenfunction $y_0 = 1$. 1

Case 2: If $\lambda < 0$, Let $\lambda = -\alpha^2$ where $\alpha > 0$. Then

$r^2 - \alpha^2 = 0 \Rightarrow r = -\alpha, +\alpha$ 1 and hence

$y(x) = c_3 x^{\alpha} + c_4 x^{-\alpha}$ 1 and $y'(x) = \alpha c_3 x^{\alpha-1} - \alpha c_4 x^{-\alpha-1}$

$y'(1) = 0 \Rightarrow \alpha c_3 - \alpha c_4 = 0 \Rightarrow \boxed{c_3 = c_4}$

$y'(2) = 0 \Rightarrow \alpha c_3 2^{\alpha-1} - \alpha c_4 2^{-\alpha-1} = 0$

(multiply by 2) $\Rightarrow \alpha c_3 2^{\alpha} - \alpha c_4 2^{-\alpha} = 0$

$(c_3 = c_4) \Rightarrow \alpha c_3 (2^{\alpha} - 2^{-\alpha}) = 0$

As $\alpha \neq 0 \Rightarrow c_3 = 0$ and so $c_4 = 0$. Hence we get trivial solution for this case. 1

Case 3: If $\lambda > 0$, Let $\lambda = \alpha^2$ where $\alpha > 0$. Then

$r^2 + \alpha^2 = 0 \Rightarrow r = \pm \alpha i$ 1 and so

$y(x) = C_5 \cos(\alpha \ln x) + C_6 \sin(\alpha \ln x)$ 1

and $y'(x) = -\frac{\alpha C_5}{x} \sin(\alpha \ln x) + \frac{\alpha C_6}{x} \cos(\alpha \ln x)$.

$y'(1) = 0 \Rightarrow \frac{\alpha C_6}{1} = 0 \Rightarrow \boxed{C_6 = 0}$

$y'(2) = 0 \Rightarrow -\frac{\alpha C_5}{2} \sin(\alpha \ln 2) = 0$

$\Rightarrow \alpha_n \ln 2 = n\pi \; ; \; n = 1, 2, \cdots$

$\Rightarrow \alpha_n = \frac{n\pi}{\ln 2}$ 1 $; \; n = 1, 2, \cdots$

So $\lambda_n = \alpha_n^2 = \frac{n^2\pi^2}{(\ln 2)^2}$ 1 $; n = 1, 2, \cdots,$ are eigenvalues

corresponding to the eigen functions

$y_n = \cos(\alpha_n \ln x)$ 1. #

6. **[9 points]** Expand $f(x) = x^2$, $0 < x < 2$, into Fourier Bessel series using Bessel functions of order 2 under the boundary condition $J_2(2\alpha) = 0$.

This is case 1 with $n=2$, $b=2$. ①

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \quad \text{where}$$

$$c_i = \frac{2}{4 J_3^2(2\alpha_i)} \int_0^2 x J_2(\alpha_i x) f(x)\,dx$$

$$= \frac{1}{2 J_3^2(2\alpha_i)} \int_0^2 x^3 J_2(\alpha_i x)\,dx \quad ①$$

Let $t = \alpha_i x$ then $dt = \alpha_i dx$ ① and so

$$c_i = \frac{1}{2 J_3^2(2\alpha_i)} \int_0^{2\alpha_i} \frac{t^3}{\alpha_i^3} J_2(t) \frac{dt}{\alpha_i} \quad ①$$

$$= \frac{1}{2\alpha_i^4 J_3^2(2\alpha_i)} \int_0^{2\alpha_i} \frac{d}{dt}\left[t^3 J_3(t)\right] dt \quad ②$$

$$= \frac{1}{2\alpha_i^4 J_3^2(2\alpha_i)} \left[t^3 J_3(t)\right]_0^{2\alpha_i}$$

$$= \frac{2^3 \alpha_i^3 J_3(2\alpha_i)}{2\alpha_i^4 J_3^2(2\alpha_i)} = \frac{4}{\alpha_i J_3(2\alpha_i)} \quad ②$$

So

$$f(x) = \sum_{i=1}^{\infty} \frac{4}{\alpha_i J_3(2\alpha_i)} J_2(\alpha_i x). \quad ① \quad \divideontimes$$

7. **[10 points]** Write the first **Two Nonzero** terms of the Fourier Legendre series of the function $f(x) = x^3, \ -1 < x < 1$.

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \quad \text{where} \quad c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x) P_n(x)\, dx$$

$$c_0 = \frac{1}{2}\int_{-1}^{1} x^3 \cdot 1 \, dx = 0. \quad 2$$

$$c_1 = \frac{3}{2}\int_{-1}^{1} x^3 \cdot x \, dx = \frac{3}{2}\cdot 2\int_{0}^{1} x^4 dx = 3\left[\frac{x^5}{5}\right]_0^1 = \frac{3}{5}. \quad 2$$

$$c_2 = \frac{5}{2}\int_{-1}^{1} x^3 \cdot \frac{1}{2}(3x^2-1)\, dx = \frac{5}{4}\int_{-1}^{1}(3x^5 - x^3)\, dx = 0 \quad 2$$

$$c_3 = \frac{7}{2}\int_{-1}^{1} x^3 \frac{1}{2}(5x^3 - 3x)\, dx \quad 1$$

$$= \frac{7}{4}\int_{-1}^{1}(5x^6 - 3x^4)\, dx$$

$$= \frac{7}{4}\cdot 2\int_{0}^{1}(5x^6 - 3x^4)\, dx$$

$$= \frac{7}{2}\left[\frac{5}{7}x^7 - \frac{3}{5}x^5\right]_0^1$$

$$= \frac{7}{2}\left[\frac{5}{7} - \frac{3}{5}\right] = \frac{7}{2}\,\frac{4}{35} = \frac{2}{5} \quad 2$$

So

$$f(x) = c_1 P_1(x) + c_3 P_3(x) + \cdots \quad 1$$

$$= \frac{3}{5}x + \frac{2}{5}\left(\frac{1}{2}(5x^3 - 3x)\right) + \cdots$$