King Fahd University of Petroleum and Minerals
Department of Mathematics

# MATH 333 - Final Exam - Term 212

Duration: 150 minutes

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Name: Answer Key    ID Number: ____________

Section Number: ____________    Serial Number: ____________

Class Time: ____________    Instructor's Name: ____________

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**Instructions:**

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 10 pages containing 6 problems and the formula sheet.

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| Question # | Points | Maximum Points |
|---|---|---|
| 1 | | 18 |
| 2 | | 22 |
| 3 | | 18 |
| 4 | | 18 |
| 5 | | 15 |
| 6 | | 14 |
| Total | | 105 |

1. **[18 points]** Solve the heat equation using separation of variables method

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < L,\ t > 0$$

subject to the conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0$$

$$u(x,0) = 3\sin(\frac{\pi x}{L}), \quad 0 < x < L.$$

Let $u(x,t) = X(x)\,T(t)$ then $X''T = XT' \Rightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda$

We deduce that $X'' + \lambda X = 0$ and $T' + \lambda T = 0$ ①

$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0$ and $u(L,t) = X(L)T(t) = 0 \Rightarrow X(L) = 0$ ①

We first solve the problem $X'' + \lambda X = 0$, $X(0) = 0$, $X(L) = 0$

The auxiliary equation is $m^2 + \lambda = 0$

Case 1: $\lambda = 0 \Rightarrow m^2 = 0 \Rightarrow m = 0, 0$ and hence $X(x) = A_1 + B_1 x$ ①

$X(0) = 0 \Rightarrow A_1 = 0$ and $X(L) = A_2 L = 0 \Rightarrow A_2 = 0$

So $X(x) \equiv 0 \Rightarrow u(x,t) \equiv 0$ trivial solution. ①

Case 2: $\lambda < 0$. Let $\lambda = -\alpha^2$; $\alpha > 0$. Then $m^2 - \alpha^2 = 0 \Rightarrow m = \pm\alpha$

So $X(x) = A_2 \cosh(\alpha x) + B_2 \sinh(\alpha x)$ ①

$X(0) = A_2 + 0 = 0 \Rightarrow A_2 = 0$ and $X(L) = 0 + B_2 \sinh(\alpha L) = 0 \Rightarrow B_2 = 0$

since $\sinh(\alpha L) \neq 0$

so $X(x) \equiv 0 \Rightarrow u(x,t) \equiv 0$ trivial solution ①

Case 3: $\lambda > 0$. Let $\lambda = \alpha^2$; $\alpha > 0$. Then $m^2 + \alpha^2 = 0 \Rightarrow m = \pm \alpha i$

$X(x) = A_3 \cos(\alpha x) + B_3 \sin(\alpha x)$ ①

$X(0) = A_3 + 0 = 0 \Rightarrow A_3 = 0$

$X(L) = 0 + B_3 \sin(\alpha L) = 0 \Rightarrow \sin(\alpha L) = 0 \Rightarrow \alpha L = n\pi$ ; $n = 1, 2, \cdots$

$\Rightarrow \alpha_n = \frac{n\pi}{L}$ ; $n = 1, 2, \cdots$

②

The eigenvalues are $\lambda_n = \alpha_n^2 = \dfrac{n^2\pi^2}{L^2}$ and the corresponding eigenfunctions (nontrivial solutions) are

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right) \quad ②$$

Now we solve the equation $T' + \lambda_n T = 0 \Rightarrow T' + \dfrac{n^2\pi^2}{L^2} T = 0$ ①

The solution is $T_n(t) = c\, e^{-\frac{n^2\pi^2}{L^2}t}$ ① ; $n = 1, 2, \dots$

Hence for each $n = 1, 2, \dots$ we have

$$u_n(x,t) = X_n(x)\,T_n(t) = A_n\, e^{-\frac{n^2\pi^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right) \quad \text{where } A_n = c B_n$$

By the superposition principle

$$u(x,t) = \sum_{n=1}^{\infty} A_n\, e^{-\frac{n^2\pi^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right) \quad ②$$ is also a solution.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= A_1 \sin\left(\frac{\pi x}{L}\right) + A_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots = 3 \sin\left(\frac{\pi x}{L}\right)$$

$\Rightarrow A_1 = 3$ and $A_n = 0$ for all $n \geq 2$. ②

The solution to the heat problem is

$$u(x,t) = 3\, e^{-\frac{\pi^2}{L^2}t} \sin\left(\frac{\pi x}{L}\right). \quad ①$$

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2. **[22 points]** Solve the Laplace equation using separation of variables method

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < 1,\ 0 < y < 1$$

subject to the conditions

$$\frac{\partial u}{\partial y}(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,1) = 0, \quad 0 < x < 1.$$

$$u(0,y) = 0, \quad u(1,y) = y, \quad 0 < y < 1$$

Let $u(x,y) = X(x)Y(y)$ then $X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$

we deduce that

$X'' + \lambda X = 0$ and $Y'' - \lambda Y = 0$ 1

$\frac{\partial u}{\partial y}(x,0) = X(x)Y'(0) = 0 \Rightarrow Y'(0) = 0$ and $\frac{\partial u}{\partial y}(x,1) = X(x)Y'(1) = 0 \Rightarrow Y'(1) = 0$ 1

$u(0,y) = X(0)Y(y) = 0 \Rightarrow X(0) = 0$ 1

we first solve the problem $Y'' - \lambda Y = 0$; $Y'(0) = 0$, $Y'(1) = 0$.

The auxiliary equation is $m^2 - \lambda = 0$.

Case 1: $\lambda = 0 \Rightarrow m^2 = 0 \Rightarrow m = 0, 0$ and hence $Y(y) = A_1 + B_1 y$ 1

$\left.\begin{matrix} Y'(0) = 0 \\ Y'(0) = 0 \end{matrix}\right\} B_1 = 0$ and so $Y(y) = A_1$ is a nontrivial solution. 1

Case 2: $\lambda > 0$. Let $\lambda = \alpha^2$; $\alpha > 0$. Then $m^2 - \alpha^2 = 0 \Rightarrow m = \pm\alpha$.

$Y(y) = A_2 \cosh(\alpha y) + B_2 \sinh(\alpha y) \Rightarrow Y'(y) = \alpha A_2 \sinh(\alpha y) + \alpha B_2 \cosh(\alpha y)$ 1

$Y'(0) = 0 + \alpha B_2 = 0 \Rightarrow B_2 = 0$

$Y'(1) = 0 \Rightarrow \alpha A_2 \sinh(\alpha) + 0 = 0 \Rightarrow A_2 = 0$ since $\sinh(\alpha) \neq 0$

so $Y(y) \equiv 0 \Rightarrow u(x,y) \equiv 0$ trivial solution 1

Case 3: $\lambda<0$. Let $\lambda=-\alpha^2$; $\alpha>0$. Then $m^2+\alpha^2=0 \Rightarrow m=\pm\alpha i$

$Y(y)=A_3\cos(\alpha y)+B_3\sin(\alpha y)$ ① $\Rightarrow Y'(y)=-\alpha A_3\sin(\alpha y)+\alpha B_3\cos(\alpha y)$

$Y'(0)=0+\alpha B_3=0 \Rightarrow B_3=0$.

$Y'(1)=-\alpha A_3\sin(\alpha)=0 \Rightarrow \sin(\alpha)=0 \Rightarrow \alpha_n=n\pi$ ② ; $n=1,2,\cdots$

The eigenvalues are $\lambda_n=-\alpha_n^2=-n^2\pi^2$ and the corresponding eigenfunctions are $Y_n(y)=A_3\cos(n\pi y)$ ②

Now we solve the problem $X''+\lambda X=0$; $X(0)=0$.

Case of $\lambda=0$: $X''=0 \Rightarrow X(x)=A_4+B_4x$.

$X(0)=A_4+0=0 \Rightarrow A_4=0$ So $X_0(x)=B_4x$ ①.

Case of $\lambda_n=-n^2\pi^2$: $X''-n^2\pi^2X=0 \Rightarrow X_n(x)=A_5\cosh(n\pi x)+B_5\sinh(n\pi x)$

$X(0)=A_5+0=0 \Rightarrow A_5=0$ so $X_n(x)=B_5\sinh(n\pi x)$ ① for $n=1,2,\cdots$

By the superposition principle we have

$$u(x,y)=X_0Y_0+\sum_{n=1}^{\infty}X_nY_n$$

$$\Rightarrow u(x,y)=A_0x+\sum_{n=1}^{\infty}A_n\cos(n\pi y)\sinh(n\pi x) \quad ②$$

$$u(1,y)=A_0+\sum_{n=1}^{\infty}A_n\sinh(n\pi)\cos(n\pi y)=y$$

This is the Fourier Cosine series for $f(y)=y$. ①

$$A_0=\frac{a_0}{2}=\int_0^1 y\,dx=\frac{y^2}{2}\Big]_0^1=\frac{1}{2}. \quad ①$$

$$A_n \sinh(n\pi) = 2\int_0^1 y\cos(n\pi y)\,dy$$

$$= \frac{2}{n\pi} y \sin(n\pi y) + \frac{2}{n^2\pi^2}\cos(n\pi y)\Big]_0^1 \quad 2$$

$$= \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\Rightarrow \quad A_n = \frac{2\left[(-1)^n - 1\right]}{n^2\pi^2 \sinh(n\pi)} \quad 1$$

The solution to the main problem is

$$u(x,y) = \frac{x}{2} + \sum_{n=1}^{\infty} \frac{2\left[(-1)^n - 1\right]}{n^2\pi^2 \sinh(n\pi)} \cos(n\pi y)\sinh(n\pi x). \quad 1$$

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3. **[18 points]** The temperature in a circular plate of radius 1 is determined from the boundary-value problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial u}{\partial t}, \quad 0 < r < 1,\ t > 0$$

$$u(1,t) = 0, \quad t > 0$$

$$u(r,0) = 5, \quad 0 < r < 1.$$

Use the method of separation of variables to solve for $u(r,t)$.

Let $u(r,t) = R(r)\,T(t)$ then $R''T + \frac{1}{r}R'T = RT'$

$\Rightarrow \frac{R'' + \frac{1}{r}R'}{R} = \frac{T'}{T} = -\lambda \Rightarrow R'' + \frac{1}{r}R' + \lambda R = 0$ and $T' + \lambda T = 0$ 2

We consider $R'' + \frac{1}{r}R' + \lambda R = 0 \Rightarrow r^2 R'' + rR' + \lambda r^2 R = 0$

This is a parametric Bessel equation of order $\nu = 0$ and $\lambda = \alpha^2$; $\alpha > 0$. The general solution to this equation is 1

$R(r) = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)$. 2 Since we need the solution to be bounded at $r=0$; we need to set $c_2 = 0$. 1 So $R(r) = c_1 J_0(\alpha r)$

$u(1,t) = 0 \Rightarrow R(1) = 0 \Rightarrow J_0(\alpha) = 0$ 2

This is case 1 of the Fourier Bessel series where $\alpha = \alpha_n > 0$ are the solutions of $J_0(\alpha_n) = 0$. for $n = 1, 2, \dots$

Now we solve $T' + \lambda_n T = 0$ for $\lambda_n = \alpha_n^2$. The solution is

$T(t) = c_3 e^{-\alpha_n^2 t}$. 2 Therefore $u_n(r,t) = A_n J_0(\alpha_n r)\, e^{-\alpha_n^2 t}$

By the superposition principle we get

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r)\, e^{-\alpha_n^2 t}$$ 2

Now $u(r,0) = 5 \Rightarrow \sum_{n=1}^{\infty} A_n J_0(\alpha_n r) = 5$

This is Fourier Bessel series (case 1) for $f(r)=5$.

$$A_n = \frac{2}{J_1^2(\alpha_n)} \int_0^1 5\, r\, J_0(\alpha_n r)\, dr \quad 1$$

$$= \frac{10}{J_1^2(\alpha_n)} \int_0^{\alpha_n} \frac{t}{\alpha_n} J_0(t) \frac{dt}{\alpha_n} \quad 2 \; ; \quad \text{Let } \alpha_n r = t$$

$$= \frac{10}{\alpha_n^2 J_1^2(\alpha_n)} \int_0^{\alpha_n} \frac{d}{dt}\left[t\, J_1(t)\right] dt \quad 1$$

$$= \frac{10}{\alpha_n^2 J_1^2(\alpha_n)} \alpha_n J_1(\alpha_n) \quad 1$$

$$= \frac{10}{\alpha_n J_1(\alpha_n)}$$

The solution to the main problem is

$$u(r,t) = \sum_{n=1}^{\infty} \frac{10}{\alpha_n J_1(\alpha_n)} J_0(\alpha_n r)\, e^{-\alpha_n^2 t}. \quad 1$$

4. **[18 points]** Solve for the steady-state temperature $u(r,\theta)$ that is determined by

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}=0, \qquad 0<r<1,\ 0<\theta<\pi$$

$$u(1,\theta)=\cos\theta, \quad 0<\theta<\pi.$$

Let $u(r,\theta)=R(r)\,\Theta(\theta)$, then

$$R''\Theta+\frac{2}{r}R'\Theta+\frac{1}{r^2}R\,\Theta''+\frac{\cot\theta}{r^2}R\,\Theta'=0$$

$$\Rightarrow \frac{r^2R''+2rR'}{R}=-\frac{\Theta''+(\cot\theta)\Theta'}{\Theta}=\lambda$$

$$\Rightarrow r^2R''+2rR'-\lambda R=0 \quad \text{--- (1)} \quad \text{and} \quad \Theta''+(\cot\theta)\Theta'+\lambda\Theta=0 \quad \text{--- (2)}$$

1 1

Let $x=\cos\theta$ then equation (2) becomes

$$(1-x^2)\frac{d^2\Theta}{dx^2}-2x\frac{d\Theta}{dx}+\lambda\Theta=0 \quad 2$$

This is the Legendre equation with $\lambda_n=n(n+1)$ and 2 the corresponding solution is $\Theta(\theta)=P_n(\cos\theta)$. 2

Now we solve $r^2R''+2rR'-n(n+1)R=0$.

The auxiliary equation of this Cauchy-Euler equation is

$$m(m-1)+2m-n^2-n=0$$

$$\Rightarrow m^2-n^2+m-n=0 \Rightarrow (m-n)(m+n+1)=0$$

$$\Rightarrow m=n \quad \text{or} \quad m=-n-1$$

So $R_n(r)=Ar^n+Br^{-n-1}=Ar^n+\dfrac{B}{r^{n+1}}$ 2

since we need the solution to be bounded when $r\to 0^+$ we set $B=0$. 2 So $R_n(r)=A_n r^n$; $n=0,1,2,\ldots$

The general solution is

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) \quad 2$$

$$u(1,\theta) = \cos\theta \Rightarrow \sum_{n=0}^{\infty} A_n P_n(\cos\theta) = \cos\theta = P_1(\cos\theta)$$

$$\Rightarrow A_0 P_0(\cos\theta) + A_1 P_1(\cos\theta) + \cdots = P_1(\cos\theta)$$

$\Rightarrow A_1 = 1$ 3 and $A_n = 0$ for $n \geq 0$ except $n = 1$.

Hence the solution to the main problem is

$u(r,\theta) = r\cos\theta$. #

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5. **[15 points]** The displacement of a semi-infinite elastic string is determined from

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad x > 0,\ t > 0$$

$$u(0,t) = (1 - \mathscr{U}(t-\pi))\sin t, \quad \lim_{x\to\infty} u(x,t) = 0, \quad t > 0$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad x > 0,$$

where $\mathscr{U}$ is the unit step function. Use the Laplace transform to solve for $u(x,t)$.

$$\frac{d^2U}{dx^2} = s^2 U(x,s) - s\,u(x,0) - u_t(x,0)$$

$$\Rightarrow \frac{d^2U}{dx^2} - s^2 U = 0 \;②\; \Rightarrow U(x,s) = Ae^{sx} + Be^{-sx} \;①$$

$$\lim_{x\to\infty} u(x,t) = 0 \Rightarrow \lim_{x\to\infty} U(x,s) = 0 \Rightarrow A = 0 \;①$$

So $U(x,s) = B e^{-sx}$

$$u(0,t) = (1 - \mathscr{U}(t-\pi))\sin t = \sin t - \mathscr{U}(t-\pi)\sin t.$$

$$\xrightarrow{\mathcal{L}} U(0,s) = \frac{1}{s^2+1} \;①\; - e^{-\pi s}\,\mathcal{L}\{\sin(t+\pi)\} \;①\; ①$$

$$= \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} \;①\; = Be^0 = B$$

$$\text{So } U(x,s) = \left(\frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}\right)e^{-sx} \;②\; = \frac{e^{-sx}}{s^2+1} + \frac{e^{-(\pi+x)s}}{s^2+1}$$

$$\xrightarrow{\mathcal{L}^{-1}} u(x,t) = \sin(t-x)\,\mathscr{U}(t-x) + \sin(t-x-\pi)\,\mathscr{U}(t-x-\pi). \quad \#$$

① ① ① ① ①

6. **[14 points]** Use an appropriate Fourier transform to solve the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < \pi,\ y > 0$$

$$u(0, y) = 0, \quad u(\pi, y) = e^{-y}, \quad y > 0$$

$$u(x, 0) = 0, \quad 0 < x < \pi.$$

Hint: You may need one of the following Fourier transforms

$$\mathcal{F}_s\{e^{-bx}\} = \frac{\alpha}{b^2 + \alpha^2}, \quad \mathcal{F}_c\{e^{-bx}\} = \frac{b}{b^2 + \alpha^2}, \quad \text{where } b > 0.$$

We apply Fourier Sine transform w.r.t. $y$ 1

$$\frac{d^2U}{dx^2} - \alpha^2 U(x,\alpha) + \alpha\, u(x,0) = 0$$

$$\Rightarrow \frac{d^2U}{dx^2} - \alpha^2 U = 0 \quad 2$$

$$\Rightarrow U(x,\alpha) = A\cosh(\alpha x) + B\sinh(\alpha x) \quad 2$$

$$\mathcal{F}_s\{u(0,y)\} = 0 \Rightarrow U(0,\alpha) = 0 \Rightarrow A = 0 \quad 1$$

$$\mathcal{F}_s\{u(\pi,y)\} = \mathcal{F}_s\{e^{-y}\} \Rightarrow U(\pi,\alpha) = \frac{\alpha}{1+\alpha^2} = B\sinh(\alpha\pi) \quad 2$$

$$\Rightarrow B = \frac{\alpha}{(1+\alpha^2)\sinh(\alpha\pi)} \quad 2$$

$$\text{So } U(x,\alpha) = \frac{\alpha}{(1+\alpha^2)\sinh(\alpha\pi)}\sinh(\alpha x) \quad 2$$

$$\Rightarrow u(x,y) = \frac{2}{\pi}\int_0^\infty \frac{\alpha\sinh(\alpha x)}{(1+\alpha^2)\sinh(\alpha\pi)}\sin(\alpha y)\,d\alpha. \quad 2$$

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