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King Fahd University of Petroleum & Minerals
Department of Mathematics
Math 333 Major Exam I
The Third Semester of 2021-2022 (213)

<u>Time Allowed</u>: 90 Minutes

Name: Key ID#: ____________

Section/Instructor: ____________ Serial #: ____________

- Mobiles, calculators and smart devices are not allowed in this exam.
- Write neatly and legibly. You may lose points for messy work.
- **Show all your work. No points for answers without justification.**

Question #	Marks	Maximum Marks
1		12
2		08
3		16
4		12
5		16
6		20
7		16
Total		100

Q:1 (08 + 04 points) (a) Find the parametric equations of the tangent line of the vector function $\mathbf{r}(\mathbf{t}) = < t^2,\ 2\sin(t),\ 2\cos(t) >$ at $t = \frac{\pi}{3}$.

$r'(t) = \langle 2t, 2\cos t, -2\sin t \rangle$

$r'(\frac{\pi}{3}) = \langle \frac{2\pi}{3}, 1, -\sqrt{3} \rangle$ (3)

$r(\pi/3) = \langle \frac{\pi^2}{9}, \sqrt{3}, 1 \rangle$ (2)

Parametric equations of tangent line are

$x = \frac{\pi^2}{9} + \frac{2\pi}{3} t$

$y = \sqrt{3} + t$ (3)

$z = 1 - \sqrt{3} t$

(b) Find all points on the graph of $f(x, y) = xy$, where $D_{\hat{u}} f(x, y) = \sqrt{5}$, $\mathbf{u} = < 1, 2 >$.

$\nabla f = \langle y, x \rangle$ (1)

$\hat{u} = \frac{\langle 1, 2 \rangle}{\sqrt{5}}$ (1)

$D_{\hat{u}} f = \nabla f \cdot \hat{u}$

$\sqrt{5} = \frac{y + 2x}{\sqrt{5}}$ (1)

$\Rightarrow y + 2x = 5$ (1)

Set of points $= \{ (x, y, z) : y + 2x - 5 = 0 \}$

Q:2 (8 points) Evaluate $\int_C x^2 y\,ds$ on the lower half circle $x^2+y^2=4$ oriented counter-clockwise $(\pi < \theta < 2\pi)$.

$x = 2\cos t \Rightarrow dx = -2\sin t\,dt$ 2

$y = 2\sin t \Rightarrow dy = 2\cos t\,dt$

$$\int_C x^2 y\,ds = \int_{\pi}^{2\pi} (2\cos t)^2 (2\sin t)\sqrt{4\sin^2 t + 4\cos^2 t}\,dt$$ 2

$$= \int_{\pi}^{2\pi} 2^4 \cos^2 t \sin t\,dt$$ 1

Let $\cos t = u$

$-\sin t\,dt = du$

$$= 16\int_{-1}^{1} -u^2\,dt$$ 1

$$= -16\left[\frac{u^3}{3}\right]_{-1}^{1}$$ 1

$$= -16\left[\frac{1}{3}+\frac{1}{3}\right] = -\frac{32}{3}$$ 1

Q:3 (16 points) Consider the vector field $\mathbf{F} = \langle yz, xz + 2y, xy + 1 \rangle$ on a certain region of space.

(a) Verify that $\mathbf{F}$ is a conservative vector field.

(b) Find a potential function for $\mathbf{F}$.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz+2y & xy+1 \end{vmatrix} \quad 3$$

$$= \langle x - x, -(y-y), z - z \rangle = 0 \quad 3$$

Therefore, the vector field $\vec{F}$ is conservative. For the potential

$$u(x,y,z) = \int P(x,y,z)\,dx + G(y,z) = \int yz\,dx + G(y,z) \quad 2$$

$$= xyz + G(y,z) \quad 1$$

$$\frac{\partial u}{\partial y} = xz + G'_y(y,z) = xz + 2y \quad 1$$

$$\Rightarrow G'_y(y,z) = 2y \quad 1$$

$$\Rightarrow G(y,z) = \int 2y\,dy + H(z)$$

$$= y^2 + H(z) \quad 1$$

Hence

$$u(x,y,z) = xyz + y^2 + H(z) \quad 1$$

$$\frac{\partial u}{\partial z} = xy + H'(z) = xy + 1 \quad 1$$

$$\Rightarrow H'(z) = 1$$

$$\Rightarrow H(z) = z + C \quad 1$$

$$\Rightarrow u(x,y,z) = xyz + y^2 + z + C. \quad 1$$

Q:4 (12 points) Use Green's theorem to evaluate $\int\int_D y^2 dxdy$, where

$$D = \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{4} + y^2 \le 1\}.$$

(**Hint:** Consider $P = 0$, $Q = xy^2$ and parametrize the curve.)

$P = 0 \quad , \quad Q = xy^2$ (2)

$\frac{\partial P}{\partial y} = 0 \quad , \quad \frac{\partial Q}{\partial x} = y^2$

Green's theorem $\quad \iint_D y^2 dx dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy$ (2)

$= \oint_C P dx + Q dy$

$= \oint_C x y^2 dy$ (2)

$C: \frac{x^2}{4} + y^2 \le 1 \quad \Rightarrow \quad x = 2\cos t, \; y = \sin t$

$\therefore \; I = \int_0^{2\pi} 2\cos t \cdot \sin^2 t \cdot \cos t \, dt = 2 \int_0^{2\pi} \cos^2 t \cdot \sin^2 t \, dt$ (2)

Having in mind that $\sin 2t = 2\sin t \cos t$, we get

$I = \frac{1}{2} \int_0^{2\pi} \sin^2 2t \, dt = \frac{1}{4} \int_0^{2\pi} (1 - \cos 4t) \, dt$ (2)

$= \frac{1}{4} \left[t - \frac{\sin 4t}{4} \right]_0^{2\pi}$ (2)

$= \frac{\pi}{2}$

Q:5 (16 points) Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 25$, where $x^2 + y^2 \leq 16$ and $z \geq 0$.

$$z = f(x,y) = \sqrt{25 - x^2 - y^2}$$

$$f_x = \frac{-x}{\sqrt{25-x^2-y^2}}, \quad f_y = \frac{-y}{\sqrt{25-x^2-y^2}} \qquad 4$$

$$\text{Surface area} = \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA \qquad 2$$

$$= \iint_R \frac{5}{\sqrt{25-x^2-y^2}}\, dA \qquad 2$$

Let $x = r\cos\theta$, $y = r\sin\theta$, $0 \leq r \leq 4$.

$$\text{Surface area} = \int_{\theta=0}^{2\pi} \int_{r=0}^{4} \frac{5}{\sqrt{25-r^2}} \cdot r\,dr\,d\theta \qquad 2 + 2$$

$$= 10\pi \int_0^4 \frac{r\,dr}{\sqrt{25-r^2}} \qquad 1$$

$$= -10\pi \sqrt{25-r^2}\,\Big|_0^4 \qquad 2$$

$$\qquad 1$$

$$= 20\pi.$$

Q:6 (20 points) Verify Stokes' theorem $\mathbf{F} = \langle 3y, -xz, yz^2 \rangle$, where S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$. (orient C to be counterclockwise when viewed from above)

Parametrization of $x^2+y^2=4$.

$x = 2\cos t,\ y = 2\sin t,\ z = 2\ ;\ 0 \le t \le 2\pi$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C 3y\,dx - xz\,dy + yz^2\,dz$$

$$= \int_0^{2\pi} 3(2\sin t)(-2\sin t)\,dt - (2\cos t)(2)\cdot(2\cos t)\,dt + 0$$

$$= -\int_0^{2\pi} (12\sin^2 t + 8\cos^2 t)\,dt$$

$$= -20\pi$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -xz & yz^2 \end{vmatrix} = \langle z^2+x,\ 0-0,\ -z-3 \rangle$$

$$\hat{n} = -\frac{\nabla(x^2+y^2-2z)}{|\nabla(x^2+y^2-2z)|} = -\frac{\langle x, y, -1\rangle}{\sqrt{x^2+y^2+1}}$$

Therefore, $$\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS = \iint_R \vec{\nabla}\times\vec{F}\cdot\vec{n}\ \frac{dx\,dy}{|\vec{n}\cdot\vec{k}|}$$

$$= -\iint_R (xz^2 + x^2 + z + 3)\,dx\,dy$$

$$= -\iint_R x\left(\frac{x^2+y^2}{2}\right)^2 + x^2 + \frac{x^2+y^2}{2} + 3\Big]\,dx\,dy$$

$$= -\int_0^{2\pi}\int_0^2 \Big[(r\cos\theta)\left(\frac{r^4}{4}\right) + r^2\cos^2\theta + \frac{r^2}{2} + 3\Big]\,r\,dr\,d\theta$$

$$= -20\pi$$

Q:7 (16 points) Use the divergence theorem to evaluate $\iint_S (\mathbf{F}.\mathbf{n})dS$, where $\mathbf{F}(x,y,z) = < y,\ x,\ z^2 >$ and D is the region in R^3 bounded by the paraboloid $z = x^2+y^2$ and the plane $z = 1$.

$$\vec{\nabla}.\vec{F} = 0 + 0 + 2z \qquad 2$$

Divergence theorem

$$\iint_S \vec{F}.\vec{n}\, dS = \iiint_D \nabla.\vec{F}\, dV \qquad 2$$

$$= \iiint_D 2z\, dV$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{r^2}^{1} 2z\, dz\,.\, r\,dr\,d\theta \qquad 3 + 3$$

$$= 2\pi \int_0^1 (1-r^4)\,.\,r\,dr \qquad 1 + 1$$

$$= 2\pi\left[\frac{r^2}{2} - \frac{r^6}{6}\right]_0^1 \qquad 2$$

$$= 2\pi\left[\frac{1}{2}-\frac{1}{6}\right] \qquad 2$$

$$= \frac{2\pi}{3}$$