Master

King Fahd University of Petroleum & Minerals
Department of Mathematics
Math 333 Major Exam II
The Third Semester of 2021-2022 (213)

Time Allowed: 100 Minutes

Name: ______________________ ID #: ______________

Section #: ________ Key Serial Number: __________

Instructions:

1. Make sure that you have **4** questions of multiple choice questions **(MCQ)** and **05 written problems**
2. **Show all your work** for written problems. No points for answers without justification.
3. **Calculators, Mobiles and Smart Devices** are not allowed.

MCQ Problems:

Circle your answer in the table below: 6 points for each correct answer

Question #	Answer					Grade
1	(a)	(b)	(c)	(d)	(e)	
2	(a)	(b)	(c)	(d)	(e)	
3	(a)	(b)	(c)	(d)	(e)	
4	(a)	(b)	(c)	(d)	(e)	
Total:						

Written Problems:

Question #	Grade	Maximum Points
5		12
6		14
7		16
8		16
9		18
Total:		76

MCQ:	V1	V2	V3	V4
1	e	b	d	a
2	a	d	b	c
3	c	d	a	b
4	d	a	b	e

V4 **Q:1** (6 points) $\mathcal{L}^{-1}\{\frac{2s+5}{s^2+6s+34}\} =$ $\frac{2(s+3)}{(s+3)^2+5^2} - \frac{1}{(s+3)^2+5^2}$

$= 2e^{-3t}\cos 5t - \frac{1}{5}e^{-3t}\sin 5t$

(a) $2\,e^{-3t}\cos(5t) - \frac{1}{5}e^{-3t}\sin(5t)$

(b) $2\,e^{-3t}\cos(5t) + \frac{1}{5}e^{-3t}\sin(5t)$

(c) $2\,e^{-3t}\cos(5t) + \frac{3}{5}e^{-3t}\sin(5t)$

(d) $2\,e^{-3t}\cos(5t) - \frac{3}{5}e^{-3t}\sin(5t)$

(e) $2\,e^{-3t}\cos(5t) - e^{-3t}\sin(5t)$

V1: $\frac{2s+1}{(s+3)^2+5^2} = \frac{2(s+3)}{(s+3)^2+5^2} - \frac{5}{(s+3)^2+5^2} = 2e^{-3t}\cos 5t - e^{-3t}\sin 5t$

V2: $\frac{2s+7}{(s+3)^2+5^2} = \frac{2(s+3)}{} + \frac{1}{(s+3)^2+5^2} = 2e^{-3t}\cos 5t + \frac{1}{5}e^{-3t}\sin 5t$

V3: $\frac{2s+3}{(s+3)^2+5^2} = \frac{2(s+3)}{(s+3)^2+5^2} - \frac{3}{(s+3)^2+5^2} = 2e^{-3t}\cos 5t - \frac{3}{5}e^{-3t}\sin 5t$

V1 **Q:2** (6 points) $\mathcal{L}^{-1}\{\frac{5s}{(s-2)^2}\} =$ $\frac{5(s-2)+10}{(s-2)^2} = \frac{5}{(s-2)} + \frac{10}{(s-2)^2}$

$= 5e^{2t} + 10t\,e^{2t}$

(a) $5\,e^{2t} + 10te^{2t}$

(b) $6\,e^{2t} + 12te^{2t}$

(c) $4\,e^{2t} + 8te^{2t}$

(d) $8\,e^{2t} + 16te^{2t}$

(e) $5\,e^{2t} + 25te^{2t}$

V2: $\frac{8s}{(s-2)^2} = \frac{8(s-2)+16}{(s-2)^2} = \frac{8}{s-2} + \frac{16}{(s-2)^2} = 8e^{2t} + 16t\,e^{2t}$

V3: $\frac{6s}{(s-2)^2} = \frac{6(s-2)+12}{(s-2)^2} = \frac{6}{(s-2)} + \frac{12}{(s-2)^2} = 6e^{2t} + 12t\,e^{2t}$

V4: $\frac{4s}{(s-2)^2} = \frac{4(s-2)+8}{(s-2)^2} = \frac{4}{(s-2)} + \frac{8}{(s-2)^2} = 4e^{2t} + 8t\,e^{2t}$

V3 **Q:3** (6 points) $\mathcal{L}\{t\int_0^t \tau e^{-\tau} d\tau\} =$ $-\frac{d}{ds}\mathcal{L}\left\{\int_0^t \tau e^{-\tau} d\tau\right\} = -\frac{d}{ds}\left[\frac{1}{s(s+1)^2}\right]$

$= \frac{3s+1}{s^2(s+1)^3}$

(a) $\frac{3s+1}{s^2(s+1)^3}$

(b) $\frac{6s+2}{s^2(s+1)^3}$

(c) $\frac{9s+3}{s^2(s+1)^3}$

(d) $\frac{12s+4}{s^2(s+1)^3}$

(e) $\frac{3s+1}{s(s+1)^3}$

(V1) $\mathcal{L}\left\{3t\int_0^t \tau e^{-\tau} d\tau\right\} = \frac{9s+3}{s^2(s+1)^3}$

V2: $\mathcal{L}\left\{4t\int_0^t \tau e^{-\tau} d\tau\right\} = \frac{12s+4}{s^2(s+1)^3}$

V4: $\mathcal{L}\left\{2t\int_0^t \tau e^{-\tau} d\tau\right\} = \frac{6s+2}{s^2(s+1)^3}$

V2 **Q:4** (6 points) The functions $f_1(x) = x$ and $f_2(x) = x^2$ are orthogonal on $[-2, 2]$. Find constants c_1 and c_2 such that $f_3(x) = x - c_1x^2 - c_2x^3$ is orthogonal to both f_1 and f_2 on the interval $[-2, 2]$.

(a) $c_1 = 0, c_2 = \frac{5}{12}$

(b) $c_1 = 0, c_2 = \frac{5}{24}$

(c) $c_1 = 0, c_2 = \frac{5}{36}$

(d) $c_1 = 0, c_2 = \frac{5}{48}$

(e) $c_1 = 0, c_2 = \frac{7}{12}$

$0 = \int_{-2}^{2} f_1 f_3\, dx = \int_{-2}^{2} (x^2 - c_1x^3 - c_2x^4)\, dx$

$0 = \frac{x^3}{3} - c_1\frac{x^4}{4} - c_2\frac{x^5}{5}\Big|_{-2}^{2} = \frac{16}{3} - 0 - \frac{64}{5}c_2$

$\Rightarrow c_2 = \frac{16}{3} \times \frac{5}{64} = \frac{5}{12}$

and

$0 = \int_{-2}^{2} f_2 f_3\, dx = \int_{-2}^{2} (x^3 - c_1x^4 - c_2x^5)\, dx$

$0 = -c_1 \frac{x^5}{5}\Big|_{-2}^{2} \Rightarrow c_1 = 0$

V1: $f_3(x) = x - c_1x^2 - 4c_2x^3$ $\Rightarrow 4c_2 = \frac{16}{3} \times \frac{5}{64} = \frac{5}{12} \Rightarrow c_2 = \frac{5}{48}$

V3: $f_3(x) = x - c_1x^2 - 2c_2x^3$ $\Rightarrow c_2 = \frac{5}{24}$

V4: $f_3(x) = x - c_1x^2 - 3c_2x^3$ $\Rightarrow c_2 = \frac{5}{36}$

Q:5 (12 points) Solve $y' + y = 1 + \int_0^t e^{(t-\tau)} y(\tau)d\tau, \quad y(0) = 0.$

Taking Laplace transform on both sides, we get

$$sY(s) - y(0) + Y(s) = \frac{1}{s} + \frac{1}{s-1}Y(s) \qquad 1 + 1 + 1$$

$$(s+1)Y(s) = \frac{1}{s} + \frac{1}{s-1}Y(s)$$

$$\left(s+1-\frac{1}{s-1}\right)Y(s) = \frac{1}{s} \qquad 1$$

$$Y(s) = \frac{s^2-1}{s(s^2-2)} \qquad 2$$

$$Y(s) = \frac{1}{s^2-2} - \frac{1}{s(s^2-2)}$$

Inverting,

$$y(t) = \frac{1}{\sqrt{2}}\sinh\sqrt{2}t - 1 * \frac{1}{\sqrt{2}}\sinh\sqrt{2}t \qquad 1 + 1$$

$$= \quad '' \quad - \frac{1}{\sqrt{2}}\int_0^t \sinh\sqrt{2}t\, dt \qquad 1$$

$$= \quad '' \quad - \frac{1}{2}\cosh\sqrt{2}t\Big|_0^t \qquad 1$$

$$= \frac{1}{\sqrt{2}}\sinh\sqrt{2}t - \frac{1}{2}\cosh\sqrt{2}t + \frac{1}{2}$$

$$1 + 1$$

Q:6 (14 points) Solve the IVP : $y'' + y' + 2y = \delta(t+3)$, $y(0) = 0$, $y'(0) = 0$.

Taking L.T. on both sides, we get

$$s^2 Y(s) - s y(0) - y'(0) + s Y(s) - y(0) + 2 Y(s) = e^{3s} \qquad 2 + 1$$

$$(s^2 + s + 2) Y(s) = e^{3s} \qquad 2$$

$$Y(s) = \frac{e^{3s}}{s^2+s+2} \qquad 1$$

Inverting $y(t) = \mathcal{L}^{-1}\left\{\frac{e^{3s}}{s^2+s+2}\right\}$

Now $\mathcal{L}^{-1}\left\{\frac{1}{s^2+s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+\frac{1}{2})^2 + \frac{7}{4}}\right\} \qquad 2 + 2$

$$= \frac{2}{\sqrt{7}} e^{-t/2} \sin \frac{\sqrt{7}}{2} t \qquad 2$$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{e^{3s}}{s^2+s+2}\right\} = \frac{2}{\sqrt{7}} e^{-\frac{1}{2}(t+3)} \sin\left(\frac{\sqrt{7}}{2}(t+3)\right) u(t+3).$$

2

Q:7 (10 + 6 = 16 points) Let $f : \mathbb{R} \to \mathbb{R}$ be the 2π-periodic function

$$f(x) = (x-\pi)^2, \quad x \in [0, 2\pi).$$

(a) Find the Fourier series of f.

(b) Deduce $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(a) As the function is even, we have $b_n = 0 \ \forall n$ 2

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} (x-\pi)^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi} y^2 dy = \frac{2\pi^2}{3}$$ 2

$$a_n = \frac{1}{\pi}\int_0^{2\pi} (x-\pi)^2 \cos(nx) dx = \frac{1}{\pi}\int_{-\pi}^{\pi} y^2 \cos(ny+n\pi) dy$$ 2

$$= \frac{(-1)^n}{\pi}\int_{-\pi}^{\pi} y^2 \cos(ny) dy$$

$$= \frac{(-1)^n}{\pi} \cdot \frac{4}{n^2}(-1)^n \pi$$ 2

$$= \frac{4}{n^2}$$ 2

Fourier series of $f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$

(b) $\pi^2 = f(0) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$ 3

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

and

$0 = f(\pi) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ 3

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

Q:8 (12 + 4 = 16 points) Let $f : \mathbb{R} \to \mathbb{R}$ be the odd 2π-periodic function

$$f(x) = x(\pi - x), \quad x \in [0, \pi].$$

(a) Find the Fourier series of f.

(b) Deduce $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n-1)^3}$.

(a) As the function is odd, we have $a_n = 0$, $n = 0, 1, 2, \ldots$ 2

2

$$b_n = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin(nx)\, dx$$

04

$$= \frac{2}{\pi} \left\{ -x(\pi - x) \frac{\cos nx}{n} \Big|_0^{\pi} + \frac{1}{n} \int_0^{\pi} (\pi - 2x) \cos(nx)\, dx \right\}$$

$$= \frac{2}{n\pi} \left\{ (\pi - 2x) \frac{\sin(nx)}{n} \Big|_0^{\pi} + \frac{2}{n} \int_0^{\pi} \sin(nx)\, dx \right\}$$

$$= \frac{4}{n^2 \pi} \left[-\frac{\cos(nx)}{n} \right]_0^{\pi}$$

02

$$= \frac{4}{n^3 \pi} \left[1 - (-1)^n \right]$$

02 $$f(x) = \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{(2n+1)^3}$$

(b) $f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4} \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}$ 4

Q:9 (14 + 2 + 2=18 points) (a) Find the eigenvalues and the eigenfunctions of the boundary-value problem:

$$x^2y'' + xy' + \lambda y = 0, \quad y(1) = 0, \quad y(5) = 0.$$

(b) Put the differential equation in self-adjoint form.
(c) Give an orthogonality relation.

(a) General solution of Cauchy – Euler DE is

$$y = c_1 \cos(\sqrt{\lambda} \ln x) + c_2 \sin(\sqrt{\lambda} \ln x)$$ 4

(i) If $\lambda \leq 0$, initial conditions imply $y = 0$. 2

(ii) For $\lambda = \alpha^2 > 0$,

$$y = c_1 \cos(\alpha \ln x) + c_2 \sin(\alpha \ln x)$$ 1

$y(1) = 0 \Rightarrow c_1 = 0$ 1

$y(5) = 0 \Rightarrow 0 = c_2 \sin(\alpha \ln 5)$

$\sin(\alpha \ln 5) = \sin(n\pi), n = 1, 2, 3, \ldots \quad (c_2 \neq 0)$

$\alpha = \dfrac{n\pi}{\ln 5}, n = 1, 2, 3, \ldots$ 2

2 + 2 Eigenvalues are $\dfrac{n^2\pi^2}{(\ln 5)^2}, n = 1, 2, \ldots$, Eigenfunctions $\sin\left(\dfrac{n\pi}{\ln 5}(\ln x)\right)$ $n = 1, 2, \ldots$

(b) $\dfrac{d}{dx}(xy') + \dfrac{\lambda}{x} y = 0$ 2

2 (c) orthogonality = $\displaystyle\int_1^5 \frac{1}{x} \sin\left(\frac{m\pi}{\ln 5} \ln x\right) \sin\left(\frac{n\pi}{\ln 5} \ln x\right) dx = 0,$ $m \neq n$