Q:1 (25 points) Use separation of variables method to solve the problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < L, \ t > 0,$$

subject to the boundary and initial conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0, t > 0$$

$$u(x,0) = f(x) = T_0 + \frac{(T_1 - T_0)x}{L}, \quad 0 < x < L.$$

Let $u(x,t) = X(x)T(t) \Rightarrow X''T = XT' \Rightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda$

$u_x(0,t) = 0 \Rightarrow X'(0)T(t) = 0 \Rightarrow X'(0) = 0$

$u_x(L,t) = 0 \Rightarrow X'(L)T(t) = 0 \Rightarrow X'(L) = 0.$

(i) $\frac{T'}{T} = -\lambda \Rightarrow T' + \lambda T = 0 \Rightarrow T(t) = c\,e^{-\lambda t}$

(ii) $\frac{X''}{X} = -\lambda \Rightarrow X'' + \lambda X = 0$, $X'(0) = 0$ and $X'(L) = 0$

(a) $\underline{\lambda = 0}$: $X'' = 0 \Rightarrow X = c_1 x + c_2 \Rightarrow X' = c_1$

$X'(0) = 0 \Rightarrow c_1 = 0$ $\boxed{X = \text{Constant}}$ $\boxed{T = \text{Constant}}$

(b) $\underline{\lambda = -\alpha^2 (\alpha \neq 0)}$: $X'' - \alpha^2 X = 0 \Rightarrow X = c_1 \cosh\alpha x + c_2 \sinh\alpha x$

$X' = +\alpha c_1 \sinh\alpha x + \alpha c_2 \cosh\alpha x$

$X'(0) = 0 \Rightarrow c_2 = 0$ and $X'(L) = 0 \Rightarrow c_1 = 0$ $\boxed{X(x) = 0 \text{ trivial solution}}$

(c) $\underline{\lambda = \alpha^2 (\alpha \neq 0)}$: $X'' + \alpha^2 X = 0 \Rightarrow X = c_1 \cos\alpha x + c_2 \sin\alpha x$

$X' = -\alpha c_1 \sin\alpha x + \alpha c_2 \cos\alpha x$

$X'(0) = 0 \Rightarrow c_2 = 0$

$X'(L) = 0 \Rightarrow c_1 \sin\alpha L = 0 \Rightarrow$ We take $c_1 \neq 0$ and $\sin\alpha L = 0$

$\Rightarrow \alpha L = n\pi \Rightarrow \alpha = \frac{n\pi}{L}$, $n = 1, 2, 3, \ldots$

$X_n(x) = c_1 \cos\frac{n\pi}{L}x$ and $T_n(t) = c\,e^{-\frac{n^2\pi^2}{L^2}t}$, $n = 1, 2, 3, \ldots$

Product solutions are

$u_n(x,t) = c_0 + c_1 \cos\frac{n\pi}{L}x \cdot c\,e^{-\frac{n^2\pi^2}{L^2}t} = A_0 + A_n e^{-\frac{n^2\pi^2}{L^2}t} \cos\frac{n\pi}{L}x$, $n = 1, 2, 3, \ldots$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2}{L^2}t} \cos\frac{n\pi}{L}x + A_0$$

Using the initial condition $u(x,0) = f(x)$

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi}{L}x, \quad 0 < x < L$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{T_1 + T_0}{2}$$

$$A_n = \frac{2}{L}\int_0^L f(x) \cos\frac{n\pi}{L}x\,dx$$

$$= 2(T_1 - T_0)\frac{\cos n\pi - 1}{n^2\pi^2} = 2(T_1 - T_0)\left(\frac{(-1)^n - 1}{n^2\pi^2}\right)$$

Hence

$$u(x,t) = \frac{T_1 + T_0}{2} + \frac{2(T_1 - T_0)}{\pi^2} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2} e^{-\frac{n^2\pi^2}{L^2}t} \cdot \cos\frac{n\pi x}{L}.$$

Q:2 (25 points) Use separation of variables method to solve the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \; 0 < x < 1, \; 0 < y < 1,$$

subject to the boundary and initial conditions

$$u(0, y) = 0, \; u(1, y) = 0, \quad 0 < y < 1,$$

$$u(x, 0) = 0, \; u(x, 1) = x, \quad 0 < x < 1.$$

Let $u(x,t) = X(x)Y(y)$. Then $\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda \Rightarrow X'' + \lambda X = 0$ and $Y'' - \lambda Y = 0$, $u(x,0)=0 \Rightarrow Y(0)=0$.

$u(0,y)=0 \Rightarrow X(0)=0$

$u(1,y)=0 \Rightarrow X(1)=0$

(i) $\underline{\lambda = 0}$: $X''=0 \Rightarrow X = C_1 + C_2 x$

$X(0)=0 \Rightarrow C_1 = 0$

$X(1)=0 \Rightarrow C_2 = 0$ $\quad \Rightarrow X(x)=0$ trivial solution

(ii) $\underline{\lambda = -\alpha^2 (\alpha \neq 0)}$: $X'' - \alpha^2 X = 0 \Rightarrow X = C_3 \cosh \alpha x + C_4 \sinh \alpha x$

$X(0)=0 \Rightarrow C_3 = 0$

$X(1)=0 \Rightarrow C_4 = 0$ $\quad \Rightarrow X(x) = 0$ trivial

(iii) $\underline{\lambda = \alpha^2 (\alpha \neq 0)}$: $X = C_5 \cos \alpha x + C_6 \sin \alpha x$

$X(0)=0 \Rightarrow C_5 = 0$ and $X(1)=0 \Rightarrow C_6 \sin\alpha = 0 \Rightarrow \alpha = n\pi$, $n=1,2,3,\dots$

$X(x) = C_6 \sin n\pi x$

$\underline{\text{Now}}$: $Y'' - \lambda Y = 0$ or $Y'' - n^2\pi^2 Y = 0$

$\Rightarrow Y = C_7 \cosh n\pi y + C_8 \sinh n\pi y$

$Y(0)=0 \Rightarrow C_7 = 0 \Rightarrow Y = C_8 \sinh n\pi y$

Product solutions are

$u_n(x,y) = A_n \sin n\pi x \sinh n\pi y, \; n = 1,2,3,\dots$

The superposition principle yields

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin n\pi x \sinh n\pi y$$

Using $u(x,1) = x \Rightarrow x = \sum_{n=1}^{\infty} A_n \sinh n\pi \sin n\pi x$

$$A_n \sinh n\pi = \frac{2}{1}\int_0^1 x \sin(n\pi x)\,dx = 2\left[\frac{-x\cos n\pi x}{n\pi}\Big|_0^1 + \int \frac{\cos n\pi x}{n\pi}dx\right]$$

$$= \frac{2}{n\pi}\left[-\cos n\pi - 0\right] + 2\frac{\sin n\pi x}{n^2\pi^2}\Big|_0^1$$

$$= \frac{-2}{n\pi}(-1)^n + 0$$

$$\Rightarrow A_n = -\frac{2(-1)^n}{n\pi \sinh n\pi}$$

Hence

$$u(x,y) = 2\sum_{n=1}^{\infty} \frac{-(-1)^n}{n\pi \sinh n\pi} \cdot \sin n\pi x \sinh n\pi y.$$

Q:3 (20 points) Use Laplace transform to solve the problem

$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 \le x \le 1, \ t > 0,$$

subject to the boundary and initial conditions

$$u(0,t) = 0, \ u(1,t) = 0, \ t > 0,$$

$$u(x,0) = 0, \ \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin(\pi x), \ 0 \le x \le 1.$$

Applying Laplace transform to both sides, we get

$$4\frac{d^2 U}{dx^2} = s^2 U(x,s) - s\,u(x,0) - u_t(x,0)$$

or

$$\frac{d^2 U}{dx^2} - \frac{s^2}{4} U = -\frac{1}{4}\sin(\pi x).$$

$$U_c = c_1 \cosh \frac{s}{2}x + c_2 \sinh \frac{s}{2} x.$$

Let $U_p = a\cos \pi x + b \sin \pi x$. Then DE

$$-a\pi^2 \cos \pi x - b\pi^2 \sin \pi x - \frac{s^2}{4} a \cos \pi x - \frac{s^2}{4} b \sin \pi x = -\frac{1}{4}\sin \pi x$$

or

$$\left(-a\pi^2 - a\frac{s^2}{4}\right)\cos \pi x + \left(-b\pi^2 - b\frac{s^2}{4}\right)\sin \pi x = -\frac{1}{4}\sin \pi x$$

$$\Rightarrow a = 0 \text{ and } b = \frac{1}{4\pi^2 + s^2}$$

$$\Rightarrow U_p = \frac{\sin \pi x}{4\pi^2 + s^2}$$

Boundary conditions: $U(0,s) = 0$ and $U(1,s) = 0$

General solution $U(x,s) = U_c + U_p$

$$= c_1 \cosh \frac{s}{2}x + c_2 \sinh \frac{s}{2}x + \frac{\sin \pi x}{4\pi^2 + s^2}$$

Using $U(0,s) = 0$ and $U(1,s) = 0$, we get

$\boxed{c_1 = 0}$ and $c_2 \sinh \frac{s}{2} = 0 \Rightarrow c_2 = 0$

Therefore,

$$U(x,s) = \frac{\sin \pi x}{4\pi^2 + s^2}$$

Inverting

$$u(x,t) = \sin(\pi x) \cdot \frac{1}{2\pi} \sin(2\pi t),$$

Q:4 (25 points) Use separation of variables to solve the problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < C, \; t > 0,$$

subject to the boundary conditions

$$u(C, t) = 0, \; t > 0,$$

$$u(r, 0) = 0, \; 0 < r < C,$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 5, \quad 0 < r < C.$$

Solution $u(r, t)$ is bounded at $r = 0$.

Let $u(r,t) = R(r)T(t)$. We find $R''T + \frac{1}{r}R'T = RT''$

$\Leftrightarrow \dfrac{R'' + \frac{1}{r}R'}{R} = \dfrac{T''}{T} = -\lambda \Leftrightarrow rR'' + R' + \lambda r R = 0, \; T'' + \lambda T = 0$

$u(C,t) = 0 \Rightarrow R(C) = 0$

$u(r,0) = 0 \Rightarrow T(0) = 0$, Thus $\begin{array}{l} rR'' + R' + \lambda r R = 0; \; R(C) = 0 \\ T'' + \lambda T = 0; \; T(0) = 0 \end{array}$

Let $\lambda = \alpha^2$. Then $rR'' + R' + \alpha^2 r R = 0$

$\Rightarrow R(r) = C_1 J_0(\alpha r)$

and $R(C) = 0 \Rightarrow J_0(\alpha C) = 0$.

α_i's are the non-zero values such that $J_0(C\alpha_i) = 0 \Rightarrow R = J_0(\alpha_i r)$

$\underline{\lambda = \alpha_i^2}$: $T'' + \alpha_i^2 T = 0 \Rightarrow T = C_2 \cos\alpha_i t + C_3 \sin\alpha_i t$

$T(0) = 0 \Rightarrow C_2 = 0$ and $T = C_3 \sin(\alpha_i t)$

Thus $u(r,t) = \sum_{i=1}^{\infty} A_i \sin(\alpha_i t) J_0(\alpha_i r)$

$\dfrac{\partial u}{\partial t} = \sum_{i=1}^{\infty} A_i \alpha_i \cos(\alpha_i t) J_0(\alpha_i r)$

Using $\left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 5$, we get $\quad 5 = \sum_{i=1}^{\infty} A_i \alpha_i J_0(\alpha_i r)$

$$\Rightarrow A_i \alpha_i = \frac{2}{C^2 J_1^2(\alpha_i C)} \int_0^C 5\, r\, J_0(\alpha_i r)\, dr$$

Let $\alpha_i r = x$

$$= \frac{10}{C^2 J_1^2(\alpha_i C)} \int_0^{\alpha_i \cdot C} \frac{1}{\alpha_i^2} x J_0(x)\, dx$$

$$= \frac{10}{C^2 \alpha_i^2 J_1^2(\alpha_i C)} \int_0^{C\alpha_i} \frac{d}{dx}\left[x J_1(x)\, dx\right]$$

$$= \frac{10}{C^2 \alpha_i^2 J_1^2(\alpha_i C)} \left[x J_1(x)\right]_0^{C\alpha_i} = \frac{10}{C \alpha_i J_1(\alpha_i C)}$$

Hence

$$u(r,t) = \frac{10}{C} \sum_{n=1}^{\infty} \frac{J_0(\alpha_i r)}{\alpha_i^2 J_1(\alpha_i C)} \sin(\alpha_i t).$$

Q:5 (25 points) Find the steady-state temperature $u(r,\theta)$ in a sphere of radius 2 by solving the problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0, \ 0 < r < 2, \ 0 < \theta < \pi,$$

$$u(2,\theta) = 1 + 2\cos(\theta).$$

$u = R(r)\Theta(\theta)$. Then $R''\Theta + \frac{2}{R}R'\Theta + \frac{1}{r^2}R\Theta'' + \frac{\cot\theta}{r^2}R\Theta' = 0$

$\frac{r^2R'' + 2rR'}{R} = -\frac{\Theta'' + \cot\theta\,\Theta'}{\Theta} = \lambda$

$\Rightarrow \Theta'' + \cot\theta \cdot \Theta' + \lambda\Theta = 0$ and $r^2R'' + 2rR' - \lambda R = 0$.

Let $x = \cos\theta$. Then

$(1-x^2)\Theta''(x) - 2x\,\Theta'(x) + \lambda\,\Theta(x) = 0$; $\lambda = n(n+1), n = 0, 1, 2, \ldots$

$\Theta_n(x) = P_n(x)$; $\Theta_n(\theta) = P_n(\cos\theta)$

$r^2R'' + 2rR' - \lambda R = 0$, $\lambda_n = n(n+1)$ Cauchy-Euler Equation

Auxiliary equation $m^2 + m - n(n+1) = 0$; $m = n, -(n+1)$

$R(r) = c_1 r^n + c_2 r^{-(n+1)}$

$u(r,\theta)$ is bounded $\Rightarrow c_2 = 0$.

$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$

$u(2,\theta) = 1 + 2\cos\theta \Rightarrow 1 + 2\cos\theta = \sum_{n=0}^{\infty} A_n 2^n P_n(\cos\theta)$.

$\Rightarrow A_n = \frac{2n+1}{2 \cdot 2^n}\int_0^{\pi}(1 + 2\cos\theta)P_n(\cos\theta)\sin\theta\,d\theta$

$= \frac{2n+1}{2^{n+1}}\int_0^{\pi} P_0(\cos\theta)P_n(\cos\theta)\sin\theta\,d\theta + \frac{2n+1}{2^{n+1}}\int_0^{\pi} 2P_1(\cos\theta)P_n(\cos\theta)\sin\theta\,d\theta$

By orthogonality $A_n = 0, n \neq 0, n \neq 1$.

$A_0 = \frac{1}{2}\int_0^{\pi} P_0(\cos\theta)P_0(\cos\theta)\sin\theta\,d\theta$

$= \frac{1}{2}\int_{-1}^{1} P_0(x)P_0(x)\,dx$

$= \frac{1}{2}\int_{-1}^{1} dx = 1$

$A_1 = \frac{3 \cdot 2}{4}\int_0^{\pi} P_1(\cos\theta)P_1(\cos\theta)\sin\theta\,d\theta$

$= \frac{3}{2}\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta$

$= -\frac{3}{2}\left[\frac{\cos^3\theta}{3}\right]_0^{\pi} = -\frac{1}{2}[-1-1]$

$= 1$

$u(r,\theta) = A_0P_0(\cos\theta) + A_1 r P_1(\cos\theta)$

$= 1 + r\cos\theta$

Q6 (20 points) Use appropriate Fourier transform to solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 2, \; y > 0,$$

subject to the conditions

$$u(0, y) = 0, \quad u(2, y) = 3e^{-y}, \; y > 0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \quad 0 < x < 2$$

Using Fourier cosine transform with respect to variable y

$$\mathcal{F}_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} + \mathcal{F}_c\left\{\frac{\partial^2 u}{\partial y^2}\right\} = 0$$

$$\frac{d^2 U}{dx^2} + \left(-\alpha^2 U(x,\alpha) - u_y(x,0)\right) = 0$$

$$\Rightarrow \frac{d^2 U}{dx^2} - \alpha^2 U = 0$$

$$U(x,\alpha) = c_1 \cosh \alpha x + c_2 \sinh \alpha x$$

Now

$$\mathcal{F}_c\{u(0,y)\} = \mathcal{F}_c\{0\} \Rightarrow U(0,\alpha) = 0 \quad \text{—— (A)}$$

$$\mathcal{F}_c\{u(2,y)\} = \mathcal{F}_c\{3e^{-y}\} \Rightarrow U(2,\alpha) = \int_0^\infty 3e^{-y} \cos \alpha y \, dy = I$$

$$I = 3e^{-y}\frac{\sin \alpha y}{\alpha}\Big|_0^\infty + \frac{3}{\alpha}\int_0^\infty e^{-y} \sin \alpha y \, dy$$

$$= \frac{3}{\alpha}\left[\frac{e^{-y}\cos \alpha y}{-\alpha}\Big|_0^\infty - \int_0^\infty \frac{e^{-y}\cos \alpha y}{\alpha}\right]$$

$$= \frac{3}{\alpha}\left[\frac{1}{\alpha} - \frac{1}{3\alpha} I\right]$$

$$= \frac{3}{\alpha^2} - \frac{1}{\alpha^2} I \quad \text{or} \quad \left(1 + \frac{1}{\alpha^2}\right) I = \frac{3}{\alpha^2}$$

$$I = \frac{3}{\alpha^2 + 1} = U(2,\alpha) \quad \text{—— (B)}$$

Conditions (A) and (B) give $\boxed{c_1 = 0}$, and $c_2 = \frac{3}{(\alpha^2+1)\sinh 2\alpha}$

Therefore

$$U(x,\alpha) = \frac{3 \sinh \alpha x}{(\alpha^2+1)\sinh 2\alpha}$$

$$\Rightarrow u(x,y) = \frac{2}{\pi}\int_0^\infty \frac{3 \sinh \alpha x}{(\alpha^2+1)\sinh 2\alpha} \cdot \cos \alpha y \, d\alpha.$$

Q:6 (20 points) Use appropriate Fourier transform to solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \; 0 < x < 2, \; y > 0,$$

subject to the conditions

$$u(0, y) = 0, \; u(2, y) = 3\, e^{-y}, \; y > 0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \; 0 < x < 2.$$

Using Fourier cosine transform with respect to variable y

$$\mathcal{F}_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} + \mathcal{F}_c\left\{\frac{\partial^2 u}{\partial y^2}\right\} = 0$$

$$\frac{d^2 U}{dx^2} + \left(-\alpha^2 U(x,\alpha) - u_y(x,0)\right) = 0$$

$$\Rightarrow \frac{d^2 U}{dx^2} - \alpha^2 U = 0$$

$$U(x,\alpha) = c_1 \cosh \alpha x + c_2 \sinh \alpha x \quad \text{— (A)}$$

Now

$$\mathcal{F}_c\{u(0,y)\} = \mathcal{F}_c\{0\} \Rightarrow U(0,\alpha) = 0$$

$$\mathcal{F}_c\{u(2,y)\} = \mathcal{F}_c\{3e^{-y}\} \Rightarrow U(2,y) = \int_0^\infty 3\, e^{-y} \cos \alpha y \, dy = I$$

$$I = 3\, e^{-y} \frac{\sin \alpha y}{\alpha}\Big|_0^\infty + \frac{3}{\alpha}\int_0^\infty e^{-y} \sin \alpha y \, dy$$

$$= \frac{3}{\alpha}\left[\frac{e^{-y}\cos \alpha y}{-\alpha}\Big|_0^\infty - \int_0^\infty \frac{e^{-y}\cos \alpha y}{\alpha}\right]$$

$$= \frac{3}{\alpha}\left[\frac{1}{\alpha} - \frac{1}{3\alpha} I\right]$$

$$= \frac{3}{\alpha^2} - \frac{1}{\alpha^2} I \quad \text{or} \quad \left(1 + \frac{1}{\alpha^2}\right) I = \frac{3}{\alpha^2}$$

$$I = \frac{3}{\alpha^2+1} = U(2,\alpha) \quad \text{— (B)}$$

Conditions (A) and (B) give $\boxed{c_1 = 0}$, and $c_2 = \frac{3}{(\alpha^2+1)\sinh 2\alpha}$

Therefore

$$U(x,\alpha) = \frac{3 \sinh \alpha x}{(\alpha^2+1)\sinh 2\alpha}$$

$$\Rightarrow u(x,y) = \frac{2}{\pi}\int_0^\infty \frac{3 \sinh \alpha x}{(\alpha^2+1)\sinh 2\alpha} \cdot \cos \alpha y \, d\alpha.$$