King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 333 Major Exam I
The First Semester of 2022-2023 (221)

Date: October 05, 2022 **Time Allowed: 120 Minutes**

Name: SOLUTION ID#: ____________

Section/Instructor: ____________ Serial #: ____________

- Mobiles, calculators and smart devices are not allowed in this exam.
- Write neatly and legibly. You may lose points for messy work.
- Show all your work. No points for answers **without justification.**

Question #	Marks	Maximum Marks
1		12
2		12
3		12
4		14
5		12
6		14
7		14
8		10
Total		100

Similar to Q:40 (9.1)

Q:1 (12 points) Given $\vec{r}''(t) = \sec^2 t\hat{\imath} + \cos t\hat{\jmath} - \sin t\hat{k}$ with $\vec{r}'(0) = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}$ and $\vec{r}(0) = -3\hat{\imath} + \hat{\jmath} - 2\hat{k}$. Find the vector function $\vec{r}(t)$.

Sol: $\vec{r}'(t) = \tan t\,\hat{\imath} + \sin t\,\hat{\jmath} + \cos t\,\hat{k} + \vec{c}$ ②

$\vec{r}'(0) = 0\hat{\imath} + 0\hat{\jmath} + 1\hat{k} + \vec{c} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}$ ①

$\Rightarrow \quad \vec{c} = 2\hat{\imath} - 3\hat{\jmath}$ ②

$\vec{r}'(t) = (\tan t + 2)\hat{\imath} + (\sin t - 3)\hat{\jmath} + \cos t\,\hat{k}$ ①

$\vec{r}(t) = (\ln \sec t + 2t)\hat{\imath} + (-\cos t - 3t)\hat{\jmath} + \sin t\,\hat{k} + \vec{D}$ ②

$\vec{r}(0) = 0\hat{\imath} - \hat{\jmath} + 0\hat{k} + \vec{D} = -3\hat{\imath} + \hat{\jmath} - 2\hat{k}$ ①

$\vec{D} = -3\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$ ②

$\vec{r}(t) = (\ln \sec t + 2t - 3)\hat{\imath} + (2 - \cos t - 3t)\hat{\jmath}$
$+ (\sin t - 2)\hat{k}$ ①

Similar to Q:31 (9.5)

Q:2 (12 points) Find the directional derivative of $f(x, y) = 3x^2 + y^2$ at $(1, 1)$ in the direction of tangent to the graph of $2x^2 + 3y^2 = 30$ at the point $(3, 2)$.

Sol:

$4x + 6y\,y' = 0 \Rightarrow y' = -\frac{2x}{3y}$ (2)

$m = y'\big|_{(3,2)} = -1$ (1)

$\vec{u} = \langle -1, 1\rangle$ OR $\langle 1, -1\rangle$

$\hat{u} = \pm \frac{1}{\sqrt{2}}\langle 1, -1\rangle$ (2)

$\nabla f = \langle 6x, 2y\rangle$ (2)

$\nabla f(1,1) = \langle 6, 2\rangle$ (2)

$D_{\hat{u}} f(1,1) = \pm \frac{6-2}{\sqrt{2}} = \pm 2\sqrt{2}$ (3)

Similar to HW

Q:3 (12 points) Find work done by the force $\vec{F}(x,y) = (5x^2 + 2y^2)\hat{\imath} - 3xy\hat{\jmath}$ acting along the curve $y = x^2$ for x from 0 to 1 and then along the curve $y = \sqrt{x}$ for x from 1 to 0.

Sol: $W = \oint_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$ 2

$C_1: y = x^2$ x from 0 to 1

$C_2: y = \sqrt{x}$ x from 1 to 0 2

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} (5x^2 + 2y^2)\,dx - 3xy\,dy$$

$$= \int_0^1 (5x^2 + 2x^4)\,dx - 3x\cdot x^2\cdot 2x\,dx$$

$$= \int_0^1 (5x^2 - 4x^4)\,dx = \frac{5}{3} - \frac{4}{5} = \frac{13}{15} \quad 3$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_2} (5x^2 + 2y^2)\,dx - 3xy\,dy$$

$$= \int_1^0 (5x^2 + 2x)\,dx - 3x\cdot\sqrt{x}\cdot\frac{1}{2\sqrt{x}}\,dx$$

$$= \int_1^0 (5x^2 + \tfrac{1}{2}x)\,dx = -\frac{1}{4} - \frac{5}{3} = \frac{-23}{12} \quad 3$$

$$W = \frac{13}{15} - \frac{23}{12} = -\frac{63}{60} \quad 2$$

Same/Similar to Q25 (9.9)

Q:4 (14 points) Let $\vec{F}(x,y,z) = (y - yz\sin x)\,\hat{\imath} + (x + z\cos x)\,\hat{\jmath} + y\cos x\,\hat{k}$ be a vector field and $\vec{r}(t) = 2t\,\hat{\imath} + (1+\cos t)^2\,\hat{\jmath} + 4\sin^3 t\,\hat{k}$, $0 \le t \le \frac{\pi}{2}$ be a curve C. Show that $\vec{F}$ is a conservative vector field. Find potential function ϕ for $\vec{F}$ and evaluate $\int_C \vec{F}\,.\,d\vec{r}$ using ϕ.

Sol: $P = y - yz\sin x$, $Q = x + z\cos x$, $R = y\cos x$

$$\frac{\partial P}{\partial y} = 1 - z\sin x = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = -y\sin x = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \cos x = \frac{\partial R}{\partial y} \qquad 3$$

$\vec{F}$ is a conservative vector field.

There exists $\phi(x,y,z)$ such that $\nabla\phi = \vec{F}$ 1

$$\frac{\partial \phi}{\partial x} = P = y - yz\sin x, \quad \frac{\partial \phi}{\partial y} = x + z\cos x, \quad \frac{\partial \phi}{\partial z} = R = y\cos x \qquad 1$$

$$\phi = xy + yz\cos x + f(y,z) \qquad 1$$

$$\frac{\partial \phi}{\partial y} = x + z\cos x + \frac{\partial f}{\partial y} = x + z\cos x \Rightarrow \frac{\partial f}{\partial y} = 0 \qquad 1$$

$$f(y,z) = g(z)$$

$$\phi = xy + yz\cos x + g(z) \qquad 1$$

$$\frac{\partial \phi}{\partial z} = y\cos x + g'(z) = y\cos x \Rightarrow g'(z) = 0 \qquad 1$$

$$g(z) = c = 0 \qquad 1$$

$$\phi(x,y,z) = xy + yz\cos x \qquad 1$$

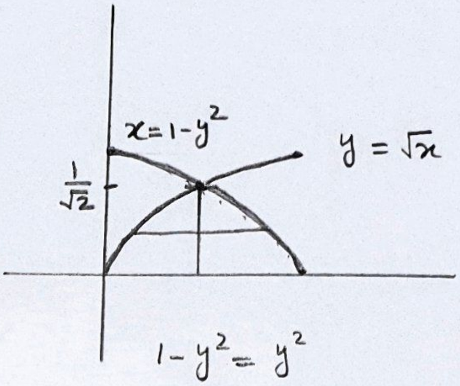
$\vec{r}(0) = \langle 0, 4, 0\rangle$ $A(0,4,0)$ 1

$\vec{r}(\frac{\pi}{2}) = \langle \pi, 1, 4\rangle$ $B(\pi,1,4)$ 1

$$\int_C \vec{F}\cdot d\vec{r} = \phi(B) - \phi(A) = \pi - 4 - 0 = \pi - 4 \qquad 1$$

Similar to Q:13 (9.12)

Q:5 (12 points) Use Green's theorem to evaluate $\oint\limits_C (y^3 - y)dx + (xy + 3xy^2)dy$, where C is the boundary of the region bounded by $y = 0$, $y = \sqrt{x}$ and $x = 1 - y^2$



Sol: $\oint\limits_C (y^3-y)dx + (xy+3xy^2)dy$

$= \iint\limits_R (y+3y^2-3y^2+1)\, dA$ ②

$= \int_0^{\frac{1}{\sqrt{2}}}\int_{y^2}^{1-y^2} (y+1)\, dx\, dy$ ②

$1-y^2 = y^2$

$2y^2 = 1$

$y = \frac{1}{\sqrt{2}}$

$= \int_0^{\frac{1}{\sqrt{2}}} (y+1)(1-y^2-y^2)\, dy$ ②

$= \int_0^{\frac{1}{\sqrt{2}}} (y-2y^3+1-2y^2)\, dy$ ②

$= \left.\frac{y^2}{2} - \frac{2y^4}{4} + y - \frac{2y^3}{3}\right|_0^{\frac{1}{\sqrt{2}}}$ ②

$= \frac{1}{4} - \frac{1}{8} + \frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}}$

$= \frac{1}{8} + \frac{3-1}{3\sqrt{2}} = \frac{1}{8} + \frac{\sqrt{2}}{3}$ ②

Similar to Q:30 (9.13)

Q:6 (14 points) Let $\vec{F}(x, y, z) = 2z\hat{k}$ be a vector field and S is part of the surface $2x^2 + 2y^2 + z = 18$ inside the cylinder $x^2 + y^2 = 9$. Find flux of $\vec{F}$ through the surface S.

Sol: $\text{Flux} = \iint_S \vec{F}\cdot\hat{n}\, ds$ ②

$g(x,y,z) = 2x^2 + 2y^2 + z - 18 = 0$

$\nabla g = \langle 4x, 4y, 1\rangle$, $\hat{n} = \dfrac{\langle 4x, 4y, 1\rangle}{\sqrt{1+16x^2+16y^2}}$ ②

$z = -2x^2 - 2y^2 + 18$

$ds = \sqrt{1+(-4x)^2+(-4y)^2}\, dA = \sqrt{1+16x^2+16y^2}\, dA$ ②

$\text{Flux} = \iint_S \dfrac{2z}{\sqrt{1+16x^2+16y^2}}\, ds = \iint_R \dfrac{2z}{\cancel{\sqrt{1+16x^2+16y^2}}} \cancel{\sqrt{1+16x^2+16y^2}}\, dA$ ②

$= \int_0^{2\pi}\int_0^3 2[18-2r^2]\, r\, dr\, d\theta$ ②

$= 2(2\pi)\left[9r^2 - \dfrac{2r^4}{4}\right]_0^3$ ②

$= 4\pi\left[81 - \dfrac{81}{2}\right] = 4\pi\left(\dfrac{81}{2}\right)$

$= 162\pi$ ②

Similar to Q:13 (9.14)

Q:7 (14 points) Use **Stokes' theorem** to evaluate $\iint\limits_S curl\vec{F}\,.\,\hat{n}\,ds$, where $\vec{F}(x,y,z) = 3yz\,\hat{\imath} + 4x\hat{\jmath} + y^2z^3e^{3x^2}\,\hat{k}$ and S is the portion of the paraboloid $z = x^2 + y^2$ for $0 \le z \le 9$ in the first octant.

Sol: Need to evaluate $\oint_C \vec{F}\cdot d\vec{r}$ 2

$C: \; x^2 + y^2 = 9 \;,\; z = 9$

$x = 3\cos\theta, \; y = 3\sin\theta, \; z = 9, \quad 0 \le \theta \le 2\pi$ 2

$\oint_C \vec{F}\cdot d\vec{r} = \oint_C^{\frac{\pi}{2}} 3yz\,dx + 4x\,dy + 0 \quad , \quad dz = 0$ 2

$= \int_0^{\frac{\pi}{2}} 3\cdot 3\sin\theta \cdot 9 \cdot (-3\sin\theta)\,d\theta + 4\cdot 3\cos\theta \cdot 3\cos\theta\,d\theta$ 2

$= \int_0^{\frac{\pi}{2}} [-243\sin^2\theta + 36\cos^2\theta]\,d\theta$ 2

$= \int_0^{\frac{\pi}{2}} \left[\frac{-243}{2}(1-\cos2\theta) + \frac{36}{2}(1+\cos2\theta)\right]d\theta$ 2

$= \frac{-243}{2}\left(\theta - \frac{\sin2\theta}{2}\right) + 18\left(1 + \frac{\sin2\theta}{2}\right)\Big|_0^{\frac{\pi}{2}}$

$= \frac{-243\pi}{4} + \frac{36\pi}{4}$ 2

$= -\frac{207\pi}{4}$

Similar to Q:3 (9.16)

Q:8 (10 points) Let $\vec{F}(x,y,z) = x^3\,\hat{\imath} + y^3\,\hat{\jmath} + z^3\,\hat{k}$ and D is the region in the upper half plane bounded by $x^2+y^2+z^2=9$. Use Divergence theorem to evaluate the flux $\iint_S (\vec{F}.\hat{n})\,ds$.

Sol: $\nabla\cdot\vec{F} = 3x^2+3y^2+3z^2$ ②

$$\iint_S \vec{F}.\hat{n}\,ds = \iiint_D \nabla\cdot\vec{F}\,dV \quad ②$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{3} 3\rho^2\cdot\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta \quad ②$$

$$= \frac{3\rho^5}{5}\Big|_0^3 \cdot (-\cos\phi)\Big|_0^{\frac{\pi}{2}} \cdot 2\pi \quad ②$$

$$= \frac{3}{5}(243)(2\pi)$$

$$= \frac{1458\pi}{5} \quad ②$$