Department of Mathematics, King Fahd University of Petroleum & Minerals, Math 533 Final Exam, 2022-2023 (221)

Q1: (a) Find $\max_{|z|\leq 1}|(z-1)(z+\frac{1}{2})|$.

(b) State the Schwarz lemma. Let f be analytic $|f(z)|\leq 3$ for all $|z|=1$ and $f(0)=0$. Can $|f'(0)|>3$? Justify your answer.

(c) Expand $f(z)=\dfrac{3z}{z^2+1}$ in Laurent series valid for $0<|z+i|<2$.

Q2a: State and prove Rouche's theorem. Use it to find the number of roots of the polynomial $p(z)=z^4+5z+3$ in the annulus $1<|z|<2$.

OR

Q2a: State the Argument principle. Use it to evaluate

$$(i)\quad \int_C \tan z\,dz, \qquad C:\{z:|z-\frac{\pi}{2}|=1\}, \qquad (ii)\quad \int_{|z|=2}\frac{z^{99}}{z^{100}+1}dz.$$

Q2b: Let $f(z)=\prod_{n=2}^{12}(z-\frac{\pi}{n})$, $z\in\mathbb{C}$ and $\gamma(t)=e^{3it}, t\in[o,2\pi]$. Compute $\int_{\gamma(t)}\dfrac{f'(z)}{f(z)}dz$.

Q3a: Show that $\int_{-\infty}^{\infty}\sin(e^x)dx=\dfrac{\pi}{2}$.

Q3b: Let C_N be the boundary of a square $\{|x|\leq n\pi, |y|\leq n\pi\}$, where N is a +ve integer, show that $\lim_{n\to\infty}\int_{C_N}\dfrac{dz}{z^3\cos z}=0$.

Q3c: Evaluate $\int_0^{2\pi}\cos(\cos\theta+i\sin\theta)d\theta$ by Gauss's Mean Value theorem.

Q4: Define a meromorphic function. State Mittag-Leffler's expansion theorem and use it to show that

$$\cot z=\frac{1}{z}+2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2\pi^2}.$$

Q5: (a)State and prove Lioville's theorem (**Do not use Cauchy's inequality**).

(b) Let $f(z)$ be a bilinear transformation such that $f(\infty)=1, f(i)=i$ and $f(-i)=-i$. Find the image of the unit disk $\{z\in\mathbb{C}:|z|<1\}$ under $f(z)$.

OR

Q5: (a) State and prove Morera's theorem.

(b) (i) Find the image of the right half plane $Re(z)\geq\dfrac{3}{2}$ under the linear transformation $w=(-1+i)z-2+3i$.
(ii) If $f(z)$ has a pole of order m at z_0, then prove that f' has a pole of order $(m+1)$ at z_0.