**King Fahd University of Petroleum & Minerals**
**Department of Mathematics**
**Math 333 Major Exam I**
**The Second Semester of 2022-2023 (222)**
**Time Allowed: 120 Minutes**

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Name: ______________________ ID#: ______________

Section/Instructor: ______________________ Serial #: ______________

Key

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- Mobiles, calculators and smart devices are not allowed in this exam.
- Write neatly and eligibly. You may lose points for messy work.
- Show all your work. No points for answers **without justification.**

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| Question # | Marks | Maximum Marks |
|---|---|---|
| 1 | | 12 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 14 |
| 5 | | 12 |
| 6 | | 14 |
| 7 | | 14 |
| 8 | | 14 |
| Total | | 100 |

**Q:1** (12 points) Find the length of the curve traced by $\mathbf{r(t)} = \langle e^t\cos(2t),\ e^t\sin(2t),\ e^t\rangle$ on the interval $0 \le t \le 3\pi$.

$$r'(t) = \langle e^t \cos 2t - 2e^t \sin 2t,\ e^t \sin 2t + 2e^t \cos 2t,\ e^t \rangle \quad 3$$

$$L = \int_0^{3\pi} \|r'(t)\|\,dt \quad 1$$

$$= \int_0^{3\pi} \sqrt{(e^t \cos 2t - 2e^t \sin 2t)^2 + (e^t \sin 2t + 2e^t \cos 2t) + (e^t)^2}\ dt \quad 2$$

$$= \int_0^{3\pi} \sqrt{e^{2t}\cos^2 2t + 4e^{2t}\sin^2 2t - 4e^{2t}\cos 2t \sin 2t + e^{2t}\sin^2 2t + 4e^{2t}\cos^2 2t + 4e^{2t}\cos 2t \sin 2t + e^{2t}}\ dt \quad 2$$

$$= \int_0^{3\pi} \sqrt{e^{2t} + 4e^{2t} + e^{2t}}\ dt \quad 2$$

$$= \int_0^{3\pi} \sqrt{6}\, e^t\, dt \quad 1$$

$$= \sqrt{6}\left[e^t\right]_0^{3\pi} = \sqrt{6}\left[e^{3\pi} - 1\right] \quad 1$$

9.1

Ex. 43

**Q:2** (10 points) If $f(x, y) = x^2 + xy + y^2 - x$, find all points where $D_{\mathbf{u}}f(x, y)$ in the direction of vector $\mathbf{v} = \langle 1,\ 1\rangle$ is $\sqrt{2}$.

$$\nabla f(x,y) = \langle 2x+y-1,\ x+2y\rangle \quad 2$$

$$\hat{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1,1\rangle}{\sqrt{2}} \quad 2$$

$$D_{\hat{u}} f(x,y) = \nabla f(x,y)\cdot \hat{u} \quad 1$$

$$\sqrt{2} = \langle 2x+y-1,\ x+2y\rangle \cdot \langle 1,1\rangle/\sqrt{2} \quad 2$$

$$2 = 2x+y-1+x+2y \quad 2$$

$$\Rightarrow 3x+3y = 3$$

$$x+y = 1 \quad 1$$

$$S = \{(x,y) : x+y=1\}$$

$$\frac{9.2}{\text{Ex. }32}$$

**Q:3** (10 points) Find the work done by $\mathbf{F} = \langle y,\ x \rangle$ along the curve $C$ traced by $\mathbf{r}(t) = \langle \cos t,\ \sin t \rangle$ from $t = 0$ to $t = \dfrac{\pi}{4}$.

$$W = \int_C \vec{F} \cdot d\vec{r} \quad 2$$

$$= \int_C \langle y, x \rangle \cdot d \langle \cos t, \sin t \rangle$$

$$= \int_{t=0}^{\pi/4} \langle \sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt \quad 4$$

$$= \int_{t=0}^{\pi/4} (-\sin^2 t + \cos^2 t) \, dt \quad 1$$

$$= \int_{t=0}^{\pi/4} \cos 2t \, dt \quad 1$$

$$= \left. \frac{\sin 2t}{2} \right]_0^{\pi/4} \quad 1$$

$$= \frac{1}{2} \left[ \sin \frac{\pi}{2} - \sin 0 \right]$$

$$= \frac{1}{2} \quad 1$$

9.8

Similar Example 6

**Q:4** (14 points) Consider the conservative vector field $\mathbf{F} = \langle e^{2z}, 3y^2, 2xe^{2z} \rangle$ on a certain region of space.

(a) Find a potential function for $\mathbf{F}$.

(b) Use the **Fundamental theorem** of line integrals to evaluate $\int_{(1,1,ln3)}^{(2,2,ln3)} \mathbf{F}.d\mathbf{r}$.

(a) Let $\phi$ be a potential function such that $\vec{F} = \nabla\phi$

$$\langle e^{2z}, 3y^2, 2xe^{2z} \rangle = \langle \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \rangle \quad 3$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = e^{2z}, \quad \frac{\partial\phi}{\partial y} = 3y^2, \quad \frac{\partial\phi}{\partial z} = 2xe^{2z}$$

Int. w.r.t. to $x$, we get

$$\phi(x,y,z) = xe^{2z} + G(y,z) \quad 1$$

$$\frac{\partial\phi}{\partial y} = 0 + G_y(y,z) = 3y^2$$

$$\Rightarrow G(y,z) = y^3 + H(z) \quad 1$$

$$\therefore \phi(x,y,z) = xe^{2z} + y^3 + H(z) \quad 1$$

$$\frac{\partial\phi}{\partial z} = 2xe^{2z} + 0 + H'(z) = 2xe^{2z} \quad 1$$

$$\Rightarrow H'(z) = 0$$

$$\Rightarrow H(z) = C \quad 1$$

$$\therefore \phi(x,y,z) = xe^{2z} + y^3 \quad 1$$

(b) $\phi(1,1,\ln 3) = 1\, e^{2\ln 3} + 1 = 1 + 9 = 10$ 1

$\phi(2,2,\ln 3) = 2\cdot 9 + 8 = 26$ 1

Fundamental theorem

$$\int_C \vec{F}\cdot d\vec{r} = \Phi(B) - \Phi(A) \quad 2$$

$$= 26 - 10$$

$$= 16 \quad 1$$

9.9

EX. 23

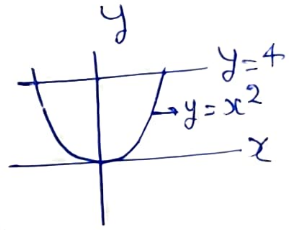
**Q:5** ( 12 points) Use Green's theorem to evaluate $\oint_C (x + y^2)dx + (2x^2 - y)dy$, where C is the boundary of the region bounded by the graphs of $y = x^2, y = 4$.

Green's theorem $\oint_C P dx + Q dy = \iint_R (Q_x - P_y) dA$ 2

Here $P = x + y^2$ $\quad Q = 2x^2 - y$ 2

$P_y = 2y$ $\quad Q_x = 4x$

$\therefore \oint_C (x+y^2)dx + (2x^2-y)dy = \iint_R (4x - 2y) dA$




$$= \int_{x=-2}^{2} \int_{y=x^2}^{4} (4x - 2y)\, dy\, dx \quad 2$$

$$= \int_{-2}^{2} \left[ 4xy - y^2 \right]_{x^2}^{4} dx \quad 1$$

$$= \int_{-2}^{2} (16x - 16 - 4x^3 + x^4)\, dx \quad 2$$

$$= \left[ 8x^2 - 16x - x^4 + \frac{x^5}{5} \right]_{-2}^{2} \quad 2$$

$$= \left[ 32 - 32 - 16 + \frac{32}{5} \right] - \left[ 32 + 32 - 16 - \frac{32}{5} \right]$$

$$= -64 + \frac{64}{5} \quad 1$$

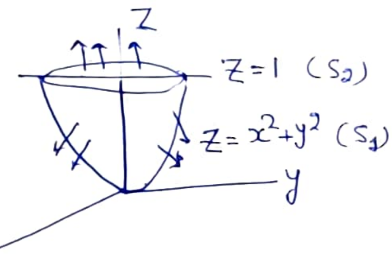
$$= \frac{-320 + 64}{5} = \frac{-256}{5}$$

9.12

Ex. 6

**Q:6** (14 points) Find the flux of $\mathbf{F} = \langle y, x^2, z\rangle$ out of the surface $S$ bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$.

**NOTE: Do not use DIVERGENCE Theorem.**




$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_{S_1} + \iint_{S_2}$$

For $S_1$: $\quad Z = x^2 + y^2$

$$Z_x = 2x, \quad Z_y = 2y$$

$$dS = \sqrt{1 + 4(x^2+y^2)}\, dA \qquad (1)$$

Let $g(x,y,z) = z - x^2 - y^2$.

$$\vec{n} = -\frac{\nabla g}{\|\nabla g\|} = \frac{\langle 2x, 2y, -1\rangle}{\sqrt{1+4(x^2+y^2)}} \qquad (2)$$

$$\iint_{S_1} = \iint_R (2xy + 2yx^2 - z)\, dA \qquad (1)$$

$$= \iint_R (2xy + 2yx^2 - (x^2+y^2))\, dA$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (2r^2\cos\theta\sin\theta + 2r^3\cos^2\theta\sin\theta - r^2)\, r\,dr\,d\theta \qquad (2)$$

$$= \int_{\theta=0}^{2\pi} \left(\frac{1}{4}\cdot 2\cos\theta\sin\theta + \frac{2}{5}\cos^2\theta\sin\theta - \frac{1}{4}\right) d\theta$$

$$= \frac{1}{4}\left[-\frac{\cos 2\theta}{2}\right]_0^{2\pi} + \frac{2}{5}\left[-\frac{\cos^3\theta}{3}\right]_0^{2\pi} - \frac{1}{4}\cdot 2\pi \qquad (1) + (1)$$

$$= \frac{1}{8}\left[-\cos 4\pi + \cos 0\right] + \frac{2}{15}\left[-\cos^3 2\pi + \cos 0\right] - \frac{\pi}{2}$$

$$= 0 + 0 - \frac{\pi}{2} = 0 - \frac{\pi}{2} \qquad (1)$$

On $S_2$:

$$Z = 1 \ ; \ dS = dA \qquad (1)$$

$$g(x,y,z) = z - 1, \quad \vec{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{\langle 0,0,1\rangle}{1} \qquad (1)$$

$$\iint_{S_2} = \iint_R z\, dA \qquad (1)$$

$$= \iint_R dA = \text{Area of the circle} \qquad (1)$$

$$= \pi$$

$$\therefore \iint_S = 0 - \frac{\pi}{2} + \pi = \frac{\pi}{2} \qquad (1)$$

$$=$$

$$\frac{9.13}{\text{Ex. }35}$$

**Q:7** ( 14 points) Use **Stokes' theorem** to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = < x+2z, 3x+y, 2y-z >$ and $C$ is the curve of intersection of the plane $x+2y+z=4$ with the coordinate planes. (Orient $C$ to be counterclockwise when viewed from above).

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\, dS$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2z & 3x+y & 2y-z \end{vmatrix}$$

$$= \langle 2, 2, 3 \rangle \qquad 3$$

$$g(x,y,z) = x+2y+z-4$$

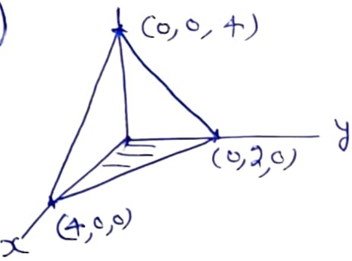
$$\vec{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{\langle 1,2,1\rangle}{\sqrt{6}} \qquad 3$$

and

$$z = 4-x-2y$$

$$z_x = -1,\ z_y = -2\ ;\ dS = \sqrt{1+1+4}\, dA \qquad 3$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (2+4+3)\, dA \qquad 2$$

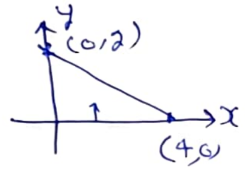



$$= 9\iint_R dA$$

$$= 9\cdot\tfrac{1}{2}\times \text{base}\times\text{height} \qquad 3$$

$$= 9\cdot\tfrac{1}{2}\times 4\times 2$$

$$= 36$$




$y = 2 - \frac{x}{2}$

or

$$3\ \left\{\begin{aligned} &= 9\int_{x=0}^{4}\int_{y=0}^{2-x/2} dy\,dx \\ &= 9\int_0^4 \left(2-\frac{x}{2}\right)dx \\ &= 9\left[2x - \frac{x^2}{4}\right]_0^4 \\ &= 9[8-4] = 36 \end{aligned}\right.$$

9.14

Ex. 8

**Q:8** ( 14 points) Let $\mathbf{F}(x, y, z) = \langle \frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2} \rangle$.
Use the **divergence theorem** to evaluate $\iint_S (\mathbf{F} \cdot \mathbf{n})dS$, where $S$ is the surface of the region bounded by the concentric spheres $x^2+y^2+z^2 = 4$ and $x^2+y^2+z^2 = 16$.

$$\frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2+z^2}\right] = \frac{(x^2+y^2+z^2)\cdot 1 - x(2x)}{(x^2+y^2+z^2)^2}$$

$$\nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle \frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2} \rangle$$

$$= \frac{-x^2+y^2+z^2 - y^2+x^2+z^2 - z^2+x^2+y^2}{(x^2+y^2+z^2)^2} \quad 3$$

$$= \frac{1}{x^2+y^2+z^2} \quad 2$$

$$\iint_S (\vec{F}\cdot\vec{n})dS = \iiint_D (\nabla\cdot\vec{F})dV$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=2}^{4} \frac{1}{\rho^2}\cdot\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \quad 3+1+2$$

$$= 2\pi(4-2)\int_0^{\pi} \sin\phi \, d\phi \quad 2$$

$$= 4\pi\left[-\cos\phi\right]_0^{\pi}$$

$$= 4\pi\left[-\cos\pi + \cos 0\right] \quad 1$$

$$= 8\pi$$

9.16

Similar Ex. 6