King Fahd University of Petroleum & Minerals
Department of Mathematics
Math 333 Major Exam II
The Second Semester of 2022-2023 (222)
Time Allowed: 120 Minutes

Name: Key ID#:

Section/Instructor: Serial #:

- Mobiles, calculators and smart devices are not allowed in this exam.
- Write neatly and eligibly. You may lose points for messy work.
- Show all your work. No points for answers without justification.

Question #	Marks	Maximum Marks
1		14
2		18
3		12
4		10
5		10
6		12
7		12
8		12
Total		100

Q:1 (8 + 6 = 14 points) (a) Use the definition of the Laplace transfor to find $\mathcal{L}\{t^2\, e^{-2t}\}$.

(b) Evaluate $\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^2+2}\right\}$

(a)
Sec 4.1, E.14

$$\mathcal{L}\{t^2 e^{-2t}\} = \int_0^\infty e^{-st}\, t^2 e^{-2t}\, dt$$

$$= \int_0^\infty t^2 e^{-(s+2)t}\, dt \quad 1$$

$$= \left.\frac{t^2 e^{-(s+2)t}}{-(s+2)}\right|_0^\infty + \int_0^\infty \frac{2t\, e^{-(s+2)t}}{(s+2)}\, dt \quad 2$$

$$= 0 - \left.\frac{2}{(s+2)^2}\, t\, e^{-(s+2)t}\right|_0^\infty + \frac{2}{s+2}\int_0^\infty \frac{e^{-(s+2)t}}{(s+2)}\, dt \quad 2$$

$$= 0 - 0 + \frac{2}{(s+2)^2}\int_0^\infty \frac{e^{-(s+2)t}}{\cancel{(s+2)}}\, dt$$

$$= \frac{2}{(s+2)^2}\left[\frac{e^{-(s+2)t}}{-(s+2)}\right]_0^\infty \quad 2$$

$$= \frac{2}{(s+2)^3},\ s > -2. \quad 1$$

Check: $\mathcal{L}\{t^2\} = \dfrac{2}{s^3}$; $\mathcal{L}\{t^2 e^{-2t}\} = \dfrac{2}{(s+2)^3}$, $s > -2$.

(b)
Sec 4.2, E16

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\} \quad 3 + 3$$

$$= \cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}\,t$$

Q:2 (6+4+8 =18 points) Find the following:

(a) $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s(s+1)}\right\}$ (b) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+2s+5}\right\}$ (c) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2s-3}\right\}$

(a) $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s} - \dfrac{e^{-s}}{s+1}\right\}$ 2

$= u(t-1) - e^{-(t-1)}\, u(t-1)$ 4

(b) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^2+2^2}\right\}$

$= \dfrac{1}{2}\, e^{-t} \sin 2t$ 4

(c) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2s-3}\right\} = \mathcal{L}^{-1}\left\{\dfrac{s}{(s-1)(s+3)}\right\}$ 2

Now,

$\dfrac{s}{(s-1)(s+3)} = \dfrac{A}{s-1} + \dfrac{B}{s+3}$

$s = A(s+3) + B(s-1)$

$s=1 \Rightarrow A = \frac{1}{4}$ 4

$s=-3 \Rightarrow B = \frac{3}{4}$

$\therefore \mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2s-3}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{4(s-1)} + \dfrac{3}{4(s+3)}\right\}$

$= \dfrac{1}{4}\mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\} + \dfrac{3}{4}\mathcal{L}^{-1}\left\{\dfrac{1}{s+3}\right\}$

$= \dfrac{1}{4} e^{t} + \dfrac{3}{4} e^{-3t}$ 2

Q:3 (12 points) Solve $f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)\, e^{t-\tau} d\tau.$

Sol: $f(t) = 3t^2 - e^{-t} - f(t) * e^{t}$ (By convolution theorem)

Taking Laplace transform on both sides, we get

$$F(s) = 3 \cdot \frac{2}{s^3} - \frac{1}{s+1} - F(s) \cdot \frac{1}{s-1} \qquad 2$$

$$\left(1 + \frac{1}{s-1}\right) F(s) = \frac{6}{s^3} - \frac{1}{s+1} \qquad 1$$

$$\frac{s}{(s-1)} F(s) = \frac{6}{s^3} - \frac{1}{s+1}$$

$$F(s) = \frac{6(s-1)}{s^4} - \frac{(s-1)}{s(s+1)} \qquad 2$$

$$= \frac{6}{s^3} - \frac{6}{s^4} - \frac{1}{s+1} + \frac{1}{s(s+1)} \qquad 2$$

$$= \frac{6}{s^3} - \frac{6}{s^4} - \frac{1}{s+1} + \left[\frac{1}{s} - \frac{1}{s+1}\right]$$

$$F(s) = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1} \qquad 2$$

Inverting,

$$f(t) = 3t^2 - t^3 + 1 - 2e^{-t}$$

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Q:4 (10 points) Solve the Initial value problem :

$$y'' + y = \delta(t - \frac{\pi}{2}) + \delta(t - \frac{3\pi}{2}), \quad y(0) = 0, \quad y'(0) = 0.$$

Sol: Let $\mathcal{L}\{y(t)\} = Y(s)$.

Taking L.T. on both sides

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\delta(t-\frac{\pi}{2})\} + \mathcal{L}\{\delta(t-\frac{3\pi}{2})\}$$

$$s^2 Y(s) - s\,\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} + Y(s) = e^{-\frac{\pi s}{2}} + e^{-\frac{3\pi s}{2}} \quad 4$$

$$(s^2+1)\,Y(s) = e^{-\frac{\pi s}{2}} + e^{-\frac{3\pi s}{2}}$$

$$Y(s) = \frac{e^{-\frac{\pi s}{2}}}{s^2+1} + \frac{e^{-\frac{3\pi s}{2}}}{s^2+1} \quad 2$$

Inverting

$$y(t) = \underbrace{\sin(t-\frac{\pi}{2})\,u(t-\frac{\pi}{2})}_{2} + \underbrace{\sin(t-\frac{3\pi}{2})\,u(t-\frac{3\pi}{2})}_{2}$$

Q:5 (10 points) Show that the set $\{\sin(x), \sin(3x), \sin(5x), \cdots\}$ is orthogonal on $[0, \frac{\pi}{2}]$. Also, find the **norm** of each function in the set.

Sol:

$$f_n(x) = \sin(2n+1)x \quad , n = 1, 2, 3, \ldots$$

$$f_m(x) = \sin(2m+1)x \quad , m = 1, 2, 3, \ldots$$

For $m \neq n$:

$$(f_n, f_m) = \int_0^{\pi/2} \sin(2n+1)x \, \sin(2m+1)x \, dx \quad 1$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[\cos 2(n-m)x - \cos 2(n+m+1)x\right] dx \quad 1$$

$$= \left[\frac{1}{4(n-m)} \sin 2(n-m)x - \frac{1}{4(n+m+1)} \sin 2(n+m+1)x\right]_0^{\pi/2} \quad 2$$

$$= 0 \quad 1$$

For $m = n$:

$$\|\sin(2n+1)\|^2 = \int_0^{\pi/2} \sin^2(2n+1)x \, dx \quad 1$$

$$= \int_0^{\pi/2} \left[\frac{1}{2} - \frac{1}{2}\cos 2(2n+1)x\right] dx \quad 1$$

$$= \left[\frac{1}{2}x - \frac{1}{4(2n+1)} \sin 2(2n+1)x\right]_0^{\pi/2} \quad 2$$

$$= \frac{\pi}{4} \quad 1$$

$$\|\sin(2n+1)\| = \sqrt{\frac{\pi}{4}}.$$

Q:6 (12 points) Find the Fourier series of the function $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 2, & 0 \leq x < \pi. \end{cases}$

Sol:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} -1\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2\,dx$$

$$= \frac{1}{\pi}\left[-x\right]_{-\pi}^{0} + \frac{1}{\pi}\left[2x\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[-\pi\right] + \frac{2\pi}{\pi} = -1+2 = 1$$

2

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} -\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2\cos nx\,dx$$

$$= \frac{1}{\pi}\left[-\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{\sin nx}{n}\right]_{0}^{\pi}$$

4

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} -\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_{0}^{\pi}$$

5

$$= \frac{1}{n\pi}\left[1-(-1)^n\right] + \frac{2}{n\pi}\left[-(-1)^n+1\right]$$

$$= \frac{3}{n\pi}\left[1-(-1)^n\right]$$

Hence

$$f(x) = \frac{1}{2} + \frac{3}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n}\sin nx.$$

1

Q:7 (12 points) Expand $f(x) = x|x|, \quad -1 < x < 1$ in an appropriate cosine or sine series.

Sol:

$f(x)$ is an odd function $\Rightarrow$ Fourier Sine Series $(a_0 = 0, a_n = 0)$ 2

$$b_n = 2\int_0^1 x^2 \sin n\pi x \, dx \quad 2$$

$$= 2\left(-\frac{x^2 \cos n\pi x}{n\pi}\right)_0^1 + \frac{4}{n\pi}\int_0^1 x \cos n\pi x \, dx \quad 2$$

$$= -\frac{2(-1)^n}{n\pi} + \frac{4}{n\pi}\left[\frac{x \sin n\pi x}{n\pi}\right]_0^1 - \frac{4}{n\pi}\int_0^1 \frac{\sin n\pi x}{n\pi} dx \quad 1$$

$$= \frac{2(-1)^{n+1}}{n\pi} + 0 - \frac{4}{n^2\pi^2}\int_0^1 \sin n\pi x \, dx \quad 2$$

$$= \;'' \; + \frac{4}{n^2\pi^2}\left[\frac{\cos n\pi x}{n\pi}\right]_0^1$$

$$= \frac{2(-1)^{n+1}}{n\pi} + \frac{4}{n^3\pi^3}\left[(-1)^n - 1\right] \quad 2$$

Hence

$$f(x) = \sum_{n=1}^{\infty}\left\{\frac{2(-1)^{n+1}}{n\pi} + \frac{4}{n^3\pi^3}\left[(-1)^n - 1\right]\right\} \sin n\pi x \quad 1$$

Q:8 (12 points) Find the eigenvalues and the eigenfunctions of the **Sturm-Liouville problem:**

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\frac{\pi}{4}) = 0.$$

(i) $\lambda = 0$: $y'' = 0$

$y = Ax + B$

$y(0) = 0 \Rightarrow B = 0$

$y(\frac{\pi}{4}) = 0 \Rightarrow A = 0$ $\Rightarrow y = 0$ trivial 2

(ii) $\lambda = -\alpha^2 < 0$: $y'' - \alpha^2 y = 0$ 2

$\Rightarrow y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$

$y(0) = 0 \Rightarrow 0 = c_1 \cosh 0 + c_2 \sinh 0$

$\Rightarrow c_1 = 0$ 1

$y(\frac{\pi}{4}) = 0 \Rightarrow 0 = c_2 \sinh \frac{\alpha \pi}{4} \Rightarrow c_2 = 0$

(iii) $\lambda = \alpha^2 > 0$: $y'' + \alpha^2 y = 0$ 2

$\Rightarrow y = c_1 \cos \alpha x + c_2 \sin \alpha x$

$y(0) = 0 \Rightarrow c_1 \cos 0 + c_2 \sin 0 = 0 \Rightarrow c_1 = 0$ 1

$y(\frac{\pi}{4}) = 0 \Rightarrow 0 = c_2 \sin \frac{\alpha \pi}{4}$

$\sin \frac{\alpha \pi}{4} = 0$ $(c_2 \neq 0)$

$\Rightarrow \frac{\alpha \pi}{4} = n\pi$ 2

$\Rightarrow \alpha = 4n, n = 1, 2, 3, \ldots$

Eigenvalues are $\lambda = 16n^2, n = 1, 2, 3, \ldots$ 1

Eigenfunctions are $y = c \sin(4nx), n = 1, 2, 3, \ldots$ 1