Q:1 (25 points) Use separation of variables method to solve the problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < \pi, \; t > 0,$$

subject to the boundary and initial conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=\pi} = 0, t > 0$$

$$u(x, 0) = x, \; 0 < x < \pi.$$

Let $u(x,T) = X(x)\,T(t) \Rightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda$ (Say) 3

$u_x(0,t) = 0 \Rightarrow X'(0) = 0$ 2

$u_x(\pi,t) = 0 \Rightarrow X'(\pi) = 0.$

(i) $\frac{T'}{T} = -\lambda \Rightarrow T' + \lambda T = 0 \Rightarrow T(t) = c\,e^{-\lambda t}$ 1

(ii) $X'' + \lambda X = 0$, $X'(0) = 0$ and $X'(\pi) = 0$ 1

(a) $\underline{\lambda = 0}$: $X'' = 0 \Rightarrow X = c_1 x + c_2$

$X' = c_1$

$X'(0) = 0 \Rightarrow c_1 = 0$ $\boxed{X = \text{Constant}}$ $\boxed{T = \text{constant}}$ ·5

2 + 1

(b) $\underline{\lambda = -\alpha^2 < 0}$: $X'' - \alpha^2 X = 0$

$\Rightarrow X = c_1 \cosh\alpha x + c_2 \sinh\alpha x$

$X' = c_1 \alpha \sinh\alpha x + \alpha c_2 \cosh\alpha x$

$X'(0) = 0 \Rightarrow c_2 = 0$ and $X'(\pi) = 0 \Rightarrow c_1 = 0$ $\boxed{X(x) = 0 \text{ Trivial}}$ 2

(c) $\underline{\lambda = \alpha^2 > 0}$: $X'' + \alpha^2 X = 0 \Rightarrow X = c_1 \cos\alpha x + c_2 \sin\alpha x$

$X' = -\alpha c_1 \sin\alpha x + \alpha c_2 \cos\alpha x$

$X'(0) = 0 \Rightarrow c_2 = 0$

$X'(\pi) = 0 \Rightarrow c_1 \alpha \sin\alpha\pi = 0 \Rightarrow \sin\alpha\pi = \sin n\pi, n = 1, 2, 3 \ldots \; (c_1 \neq 0)$ 2

$\Rightarrow \boxed{\alpha = n}$

$\therefore X_n = c_n \cos nx, \; T_n = c\,e^{-n^2 t}$ 2

The general Solution is given by 2

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-n^2 t} \cos nx.$$

Using $u(x,0) = x$, we get 2

$$A_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi}{2}$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx\,dx = \frac{2\left[(-1)^n - 1\right]}{n^2 \pi}$$ 3

Hence

$$u(x,t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\left[(-1)^n - 1\right]}{n^2 \pi} e^{-n^2 t} \cos nx$$ 2

Q:2 (20 points) Use separation of variables method to solve the problem

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ 0<x<1,\ 0<y<1,$$

subject to the boundary and initial conditions

$$u(0,y)=0,\ u(1,y)=0,\ \ 0<y<1,$$

$$u(x,0)=0,\ u(x,1)=1,\ \ 0<x<1.$$

4 Let $u(x,y)=X(x)\,Y(y)$. Then $\frac{X''}{X}=-\frac{Y''}{Y}=-\lambda \Rightarrow X''+\lambda X=0$ & $Y''-\lambda Y=0$

$u(0,y)=0 \Rightarrow X(0)=0$

$u(1,y)=0 \Rightarrow X(1)=0$ $\quad u(x,0)=0 \Rightarrow Y(0)=0.$

(i) $\lambda=0$: $X''=0 \Rightarrow X=c_1+c_2x$
$X(0)=0 \Rightarrow c_1=0$
$X(1)=0 \Rightarrow c_2=0$ $\quad\Rightarrow X(x)=$ trivial Solution 1

(ii) $\lambda=-\alpha^2<0$: $X''-\alpha^2X=0 \Rightarrow X=c_3\cosh\alpha x+c_4\sinh\alpha x$ 2
$X(0)=0 \Rightarrow c_3=0$
$X(1)=0 \Rightarrow c_4=0$ $\quad\Rightarrow X(x)=0$ Trivial.

(iii) $\lambda=\alpha^2>0$: $X=c_5\cos\alpha x+c_6\sin\alpha x$
$X(0)=0 \Rightarrow c_5=0$
$X(1)=0 \Rightarrow c_6\sin\alpha=0 \Rightarrow \alpha=n\pi,\ n=1,2,3,\ldots$ — 3

$\boxed{X(x)=c_6\sin n\pi x}$

Now $Y''-n^2\pi^2Y=0$
$\Rightarrow Y=c_7\cosh n\pi y+c_8\sinh n\pi y$
$Y(0)=0 \Rightarrow c_7=0$ $\quad\Rightarrow \boxed{Y=c_8\sinh n\pi y}$ 2

The superposition principle yields 2

$$u(x,y)=\sum_{n=1}^{\infty}A_n\sin n\pi x\,\sinh n\pi y$$

Using $u(x,1)=1$, we get 2

$$A_n\sinh n\pi=\frac{2}{1}\int_0^1 1\,\sin n\pi x\,dx$$

$$=\frac{2}{n\pi}\Big[-\cos n\pi x\Big]_0^1=\frac{2}{n\pi}\left[1-\cos n\pi\right]$$

$$=\frac{2}{n\pi}\left[1-(-1)^n\right]$$ 2

$$\Rightarrow A_n=\frac{2}{n\pi\sinh n\pi}\left[1-(-1)^n\right]$$

Hence

$$u(x,y)=2\sum_{n=1}^{\infty}\frac{1}{n\pi\sinh n\pi}\left[1-(-1)^n\right]\sin n\pi x\,\sinh n\pi y$$ 2

Q:3 (20 points) Use Laplace transform to solve the problem

$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \; x > 0, \; t > 0,$$

subject to the boundary and initial conditions

$$u(0,t) = t, \quad \lim_{x\to\infty} u(x,t) = 0, \; t > 0,$$

$$u(x,0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \; x > 0.$$

Sol:

$\mathcal{L}\left\{4\frac{\partial^2 u}{\partial x^2}\right\} = \mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\}$ Let $\mathcal{L}\{u(x,t)\} = U(x,s)$

$4\frac{d^2 U}{dx^2} = s^2 U(x,s) - s\,u(x,0) - u_t(x,0)$ 2

$= s^2 U(x,s) - 0 - 0$ 2

$\Rightarrow \frac{d^2 U}{dx^2} - \frac{s^2}{4} U = 0$ 2

G.S.: $U(x,s) = c_1 e^{-\frac{s}{2}x} + c_2 e^{+\frac{s}{2}x}$ —— (1) 2

$$\boxed{\begin{array}{l} \mathcal{L}\{\lim_{x\to\infty} u(x,t)\} = 0 \Rightarrow \lim_{x\to\infty} U(x,s) = 0 \\ \mathcal{L}\{u(0,t)\} = \mathcal{L}\{t\} \Rightarrow U(0,s) = \frac{1}{s^2} \end{array}}$$ 2

Using $\lim_{x\to\infty} U(x,s) = 0$ in (1), we get $c_2 = 0$ 2

$\therefore U(x,s) = c_1 e^{-\frac{s}{2}x}$

Using $U(0,s) = \frac{1}{s^2} \Rightarrow \boxed{\frac{1}{s^2} = c_1}$ 2

$\therefore U(x,s) = \frac{1}{s^2} e^{-\frac{s}{2}x}$ 2

Inverting

$u(x,t) = \left(t - \frac{x}{2}\right)\mathcal{U}\left(t - \frac{x}{2}\right)$ 4

Q:4 (20 points) Use separation of variables to solve the problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}, \; 0 < r < C, \; t > 0,$$

subject to the boundary conditions

$$u(C,t) = 0, \; t > 0,$$
$$u(r,0) = 0, \; 0 < r < C,$$
$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 1, \; 0 < r < C.$$

Solution $u(r,t)$ is bounded at $r = 0$.

Sol: Let $u(r,t) = R(r)T(t)$. Then $\frac{R'' + \frac{1}{r}R'}{R} = \frac{T''}{T} = -\lambda$ (2)

$\Leftrightarrow rR'' + R' + \lambda r R = 0, \; T'' + \lambda T = 0$

$u(c,t) = 0 \Rightarrow R(c) = 0$ and $u(r,0) = 0 \Rightarrow T(0) = 0$. (2)

(i) Let $\lambda = \alpha^2$. Then $rR'' + R' + \alpha^2 r R = 0$

$\Rightarrow R(r) = C_1 J_0(\alpha r)$.

and $R(c) = 0 \Rightarrow J_0(\alpha c) = 0$.

α_i's are the non-zero values such that $J_0(c\alpha_i) = 0 \Rightarrow R = J_0(\alpha_i r)$. (2)

For $\lambda = \alpha_i^2$: $T = c_2 \cos \alpha_i t + c_3 \sin \alpha_i t$

$T(0) = 0 \Rightarrow c_2 = 0$ and $T = c_3 \sin(\alpha_i t)$ (2)

Product Solution is (2)

$$u(r,t) = \sum_{i=1}^{\infty} A_i \sin(\alpha_i t) J_0(\alpha_i r)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 1 \Rightarrow 1 = \sum_{i=1}^{\infty} A_i \alpha_i \cos 0 \; J_0(\alpha_i r)$$ (2)

$$\Rightarrow A_i \alpha_i = \frac{2}{c^2 J_1^2(\alpha_i c)} \int_0^c 1 \cdot r \cdot J_0(\alpha_i r)\, dr$$ (2)

Let $\alpha_i r = x$. Then

$$A_i \alpha_i = \frac{2}{c^2 \alpha_i^2 J_1^2(\alpha_i c)} \int_0^{c\alpha_i} x J_0(x)\, dx$$

$$= \frac{2}{c^2 \alpha_i^2 J_1^2(\alpha_i c)} \int_0^{c\alpha_i} \frac{d}{dx}[x J_1(x)]\, dx$$ (2)

$$= \frac{2}{c^2 \alpha_i^2 J_1^2(\alpha_i c)} \Big[x J_1(x)\Big]_0^{c\alpha_i}$$ (2)

$$= \frac{2}{c\alpha_i J_1(\alpha_i c)}$$

Hence $$u(r,t) = \frac{2}{c} \sum_{i=1}^{\infty} \frac{J_0(\alpha_i r)}{\alpha_i J_1(\alpha_i c)} \sin(\alpha_i t).$$ (2)

Q:5 (20 points) Find the steady-state temperature $u(r,\theta)$ in a sphere of radius c by solving the problem

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}=0,\ 0<r<c,\ 0<\theta<\pi,$$

$$u(c,\theta)=\cos(\theta).$$

Let $u(r,\theta)=R(r)\,\Theta(\theta)$, Then

$$R''\Theta+\frac{2}{R}R'\Theta+\frac{1}{r^2}R\Theta'+\frac{\cot\theta}{r^2}R\Theta'=0$$

$$\Leftrightarrow \frac{r^2R''+2rR'}{R}=-\frac{\Theta''+\cot\theta\,\Theta'}{\Theta}=\lambda$$

$$\Rightarrow \Theta''+\cot\theta\,\Theta'+\lambda\Theta=0 \quad \text{and } r^2R''+2rR'-\lambda R=0 \qquad 2$$

Let $x=\cos\theta$, Then

$$(1-x^2)\Theta''_x-2x\,\Theta'_x+\lambda\,\Theta(x)=0$$

For $\lambda=n(n+1)$, $n=0,1,2,\dots$; $\Theta_n(x)=P_n(x)$; $\Theta_n(x)=P_n(\cos\theta)$. 3

Now $r^2R''+2rR'-n(n+1)R=0$ — Cauchy - Euler Equation

A.E : $m^2+m-n(n+1)=0 \Rightarrow m=n,\ -(n+1)$.

$$R(r)=C_1r^n+C_2r^{-(n+1)} \qquad 3$$

$u(r,\theta)$ is bounded $\Rightarrow C_2=0$.

Product Solution : $u(r,\theta)=\sum_{n=0}^{\infty}A_nr^nP_n(\cos\theta)$ 2

Using $u(c,\theta)=\cos\theta$, we get

$$\cos\theta=\sum_{n=1}^{\infty}A_nc^nP_n(\cos\theta)$$

$$\Rightarrow A_nc^n=\frac{2n+1}{2}\int_0^{\pi}\cos\theta\,P_n(\cos\theta)\sin\theta\,d\theta \qquad 2$$

$$\Rightarrow A_n=\frac{2n+1}{2\cdot c^n}\int_0^{\pi}P_1(\cos\theta)\,P_n(\cos\theta)\sin\theta\,d\theta \qquad 2$$

Let $\cos\theta=x$.
$-\sin\theta\,d\theta=dx$

$$=\frac{2n+1}{2\cdot c^n}\int_{-1}^{1}P_1(x)\,P_n(x)\,dx \qquad 2$$

By orthogonality $A_n=0$, $n\neq 1$.

$$A_1=\frac{3}{2c}\int_{-1}^{1}P_1(x)\,P_1(x)\,dx=\frac{3}{2c}\int_{-1}^{1}x^2dx$$

$$=\frac{3}{2c}\left[\frac{2}{3}\right]=\frac{1}{c} \qquad 2$$

Hence

$$u(r,\theta)=A_1\,r\,P_1(\cos\theta)$$

$$=\frac{1}{c}\cdot r\cdot\cos\theta \qquad 2$$

or

$$A_1=\frac{3}{2c}\int_0^{\pi}\cos^2\theta\,\sin\theta\,d\theta$$

$$=-\frac{3}{2c}\left[\frac{\cos^3\theta}{3}\right]_0^{\pi}$$

$$=\frac{1}{c}$$

Q:6 (10 points) If $\mathbf{F} = \langle x^2y^3 - z^4, 4x^5y^2z, y^4z^6 \rangle$, find (i) curl $\mathbf{F}$ and (ii) div $\mathbf{F}$.

(i)
$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3-z^4 & 4x^5y^2z & y^4z^6 \end{vmatrix} \quad 3$$

$$= \langle 4y^3z^6 - 4x^5y^2, -4z^3, 20x^4y^2z - 3x^2y^2 \rangle \quad 3$$

(ii)
$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2y^3 - z^4) + \frac{\partial}{\partial y}(4x^5y^2z) + \frac{\partial}{\partial z}(y^4z^6) \quad 1$$

$$= 2xy^3 + 8x^5yz + 6y^4z^5 \quad 3$$

Q:7 (10 points) Find the Fourier cosine integral representation of $f(x) = e^{-x}, \quad x > 0$.

$$F(\alpha) = \int_0^\infty e^{-x} \cos\alpha x \, dx$$

$$= \left. \frac{e^{-x} \sin\alpha x}{\alpha} \right|_0^\infty + \frac{1}{\alpha} \int_0^\infty e^{-x} \sin\alpha x \, dx \quad 2$$

$$= \frac{1}{\alpha} \left[\left. \frac{e^{-x} \cancel{\cos}\alpha x}{\alpha} \right|_0^\infty - \int_0^\infty \frac{e^{-x} \cos\alpha x}{\alpha} dx \right] \quad 2$$

$$= \frac{1}{\alpha^2} - \frac{1}{\alpha^2} F(\alpha) \quad 2$$

$$\left(1 + \frac{1}{\alpha^2}\right) F(\alpha) = \frac{1}{\alpha^2}$$

$$\Rightarrow F(\alpha) = \frac{1}{1+\alpha^2} \quad 2$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{1}{1+\alpha^2} \cos\alpha x \, d\alpha. \quad 2$$

Q:8 (15 points) Use Fourier sine transform to solve

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad x > 0, \; t > 0,$$

subject to the conditions

$$u(0,t) = 5, \quad t > 0,$$

$$u(x,0) = 0, \quad x > 0.$$

Sol:

$$\mathcal{F}_s\{k\, u_{xx}\} = \mathcal{F}_s\{u_t\}$$

$$-\alpha^2 k\, U(\alpha,t) + \alpha k\, u(0,t) = \frac{dU}{dt} \qquad 2$$

$$\frac{dU}{dt} + \alpha^2 k\, U = 5\alpha k \quad —① \qquad 2$$

$$U_c(\alpha,t) = C\, e^{-\alpha^2 k t} \qquad 2$$

Let $U_p = A$. Then $A = \dfrac{5\alpha k}{\alpha^2 k} = \dfrac{5}{\alpha}$ 2

General Solution of ①

$$U(\alpha,t) = C\, e^{-\alpha^2 k t} + \frac{5}{\alpha} \quad —② \qquad 2$$

Now $\mathcal{F}_s\{u(x,0)\} = 0 \Rightarrow U(\alpha,0) = 0.$ —③

Using ③ in ②, we get

$$C = -\frac{5}{\alpha} \qquad 2$$

Thus

$$U(\alpha,t) = -\frac{5}{\alpha} e^{-\alpha^2 k t} + \frac{5}{\alpha} \qquad 1$$

Inverting

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \left(-\frac{5}{\alpha} e^{-\alpha^2 k t} + \frac{5}{\alpha}\right)\sin\alpha x\, d\alpha$$

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