King Fahd University of Petroleum & Minerals
Department of Mathematics
Math 333 Major Exam I
The Second Semester of 2023-2024 (232)

Time Allowed: 120 Minutes

Name: ______________________ ID#: ______________

Section/Instructor: ______________ Serial #: ______________

Key

- Mobiles, calculators and smart devices are not allowed in this exam.
- Write neatly and eligibly. You may lose points for messy work.
- Show all your work. No points for answers **without justification.**

Question #	Marks	Maximum Marks
1		16
2		8
3		8
4		14
5		12
6		14
7		14
8		14
Total		100

Q:1 (08 + 08 points) (a) Find a vector function $\mathbf{r(t)}$ that satisfies the conditions:

$$\mathbf{r'(t)} = \langle 6, 6t, 3t^2 \rangle, \quad \mathbf{r(0)} = \langle 1, -2, 1 \rangle.$$

(b) Find the length of the curve traced by $\mathbf{r(t)} = \langle 2\cos(t),\ 2\sin(t), 3t \rangle$ on the interval $0 \le t \le 3\pi$.

Sol. 1(a)

$$r(t) = \int r'(t)\,dt$$

$$= \int \langle 6, 6t, 3t^2 \rangle dt$$

$$= \langle 6t + c_1,\ 3t^2 + c_2,\ t^3 + c_3 \rangle$$

$$r(0) = \langle 1, -2, 1 \rangle = \langle 0 + c_1,\ 0 + c_2,\ 0 + c_3 \rangle$$

$$\Rightarrow c_1 = 1,\ c_2 = -2,\ c_3 = 1.$$

Thus $r(t) = \langle 6t + 1,\ 3t^2 - 2,\ t^3 + 1 \rangle$

(b)

$$S = \int_0^{3\pi} \| r'(t) \| \, dt$$

$$= \int_0^{3\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 9}\ dt$$

$$= \int_0^{3\pi} \sqrt{4(\cos^2 t + \sin^2 t) + 9}\ dt$$

$$= \int_0^{3\pi} \sqrt{13}\ dt$$

$$= 3\pi \sqrt{13}$$

Q:2 (08 points) Find the directional derivative of $f(x,y,z) = x^2y^2(2z+1)^2$ at the point $(1,-1,1)$ in the direction of a vector $\mathbf{v} = \langle 0,\ 3, 3\rangle$

Sol:

$$\nabla f(x,y,z) = \langle 2xy^2(2z+1)^2,\ 2x^2y(2z+1)^2,\ 2x^2y^2.2(2z+1)\rangle$$

$$\nabla f(1,-1,1) = \langle 18, -18, 12\rangle$$

$$\text{unit vector} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 0,3,3\rangle}{\sqrt{18}} = \frac{\langle 0,1,1\rangle}{\sqrt{2}}$$

$$D_{\hat{u}} f = \nabla f(1,-1,1)\cdot \hat{u} = \langle 18,-18,12\rangle \cdot \frac{\langle 0,1,1\rangle}{\sqrt{2}} = -\frac{6}{\sqrt{2}} = -3\sqrt{2}$$

Q:3 (08 points) Evaluate $\int_C xydy - y^2dx$, where C is given by $x = 2t, y = 3t^3,\ 0 \le t \le 3$.

$$x = 2t, \quad y = 3t^3$$

$$dx = 2dt \quad dy = 9t^2dt$$

$$\int_C xy\,dy - y^2\,dx$$

$$= \int_{t=0}^{3} \left[(2t)(3t^3).9t^2 - 18t^6\right] dt$$

$$= \int_{t=0}^{3} 36\,t^6\,dt$$

$$= 36\,\frac{t^7}{7}\Big|_0^3$$

$$= \frac{36}{7}.3^7.$$

Q:4 (14 points) Consider the conservative vector field $\mathbf{F} = \langle e^{2z}, 3y^2, 2xe^{2z} \rangle$ on a certain region of space.

(a) Find a potential function for $\mathbf{F}$.

(b) Use the **Fundamental theorem** of line integrals to evaluate $\int_{(1,1,ln3)}^{(2,2,ln3)} \mathbf{F}.d\mathbf{r}$.

(a) Let ϕ be a potential function such that $\vec{F} = \nabla\phi$

$$\langle e^{2z}, 3y^2, 2x e^{2z} \rangle = \langle \phi_x, \phi_y, \phi_z \rangle$$

$$\Rightarrow \phi_x = e^{2z}, \quad \phi_y = 3y^2, \quad \phi_z = 2x e^{2z}$$

$\downarrow$

Int. w.r. to x, we get

$$\phi(x,y,z) = x e^{2z} + G(y,z)$$

$$\frac{\partial\phi}{\partial y} = 0 + G_y(y,z) = 3y^2$$

$$\Rightarrow G_y(y,z) = 3y^2$$

$$\Rightarrow G(y,z) = y^3 + H(z)$$

$$\therefore \phi(x,y,z) = x e^{2z} + y^3 + H(z)$$

$$\phi_z = 2x e^{2z} + 0 + H'(z) = 2x e^{2z}$$

$$\Rightarrow H'(z) = 0$$

$$\Rightarrow H(z) = C$$

$$\phi(x,y,z) = x e^{2z} + y^3$$

(b) $\phi(1,1,\ln 3) = 1 \cdot e^{2\ln 3} + 1 = 1 + 9 = 10$

$\phi(2,2,\ln 3) = 2 \cdot e^{2\ln 3} + 2^3 = 18 + 8 = 26$

Fundamental theorem

$$\int_C \vec{F} \cdot d\vec{r} = \phi(B) - \phi(A)$$

$$= 26 - 10$$

$$= 16$$

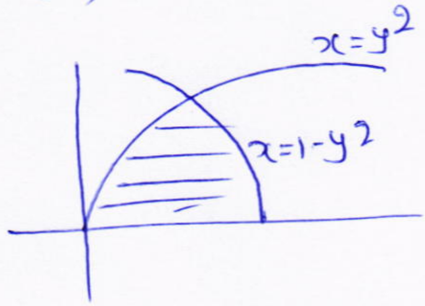
Q:5 (12 points) Use Green's theorem to evaluate $\oint_C \frac{1}{3}y^3dx+(xy+xy^2)dy$, where C is the boundary of the region in the first quadrant determined by the graphs of $y=0, x=y^2, x=1-y^2$.

Sol: Green's theorem $\oint_C P dx + Q dy = \iint_R (Q_x - P_y)\, dA$

$P = \frac{1}{3}y^3$; $Q = xy + xy^2$

$P_y = y^2$ $\quad Q_x = y + y^2$

$$\oint_C \frac{1}{3}y^3 dx + (xy+xy^2)dy = \iint_R (y+y^2-y^2)\, dA$$



$$= \iint_R y\, dA$$

$$= \int_{y=0}^{\frac{1}{\sqrt{2}}} \int_{x=y^2}^{1-y^2} y\, dx\, dy$$

$x = 1-y^2 = y^2$

$2y^2 = 1$

$y = \pm \frac{1}{\sqrt{2}}$

First quadrant $y = \frac{1}{\sqrt{2}}$

$$= \int_{y=0}^{\frac{1}{\sqrt{2}}} \left[xy \right]_{x=y^2}^{1-y^2} dy$$

$$= \int_{y=0}^{\frac{1}{\sqrt{2}}} [y - 2y^3]\, dy$$

$$= \frac{y^2}{2} - \frac{y^4}{2} \Big|_0^{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{4} - \frac{1}{8}$$

$$= \frac{1}{8}$$

Q:6 (14 points) Find the flux of $\mathbf{F} = \langle 0, 0, z \rangle$ out of the surface S bounded by the paraboloid $z = 2 - x^2 - y^2$ and the cylinder $x^2 + y^2 = 2$.

NOTE: Do not use DIVERGENCE Theorem.

Sol. Let $g(x, y, z) = x^2 + y^2 + z - 2$

$$\nabla g = \langle 2x, 2y, 1 \rangle, \quad \|\nabla g\| = \sqrt{1 + 4x^2 + 4y^2}$$

$$\vec{n} = \frac{\nabla g}{\|\nabla g\|} = \langle 2x, 2y, 1 \rangle / \sqrt{1 + 4x^2 + 4y^2}$$

$$\vec{F} \cdot \vec{n} = \frac{z}{\sqrt{1 + 4x^2 + 4y^2}}$$

Now, $z = 2 - x^2 - y^2$

$$z_x = -2x, \quad z_y = -2y$$

$$dS = \sqrt{1 + 4(x^2 + y^2)}\, dA$$

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, dS$$

$$= \iint_R z\, dA$$

$$= \iint_R (2 - x^2 - y^2)\, dA$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{2}} (2 - r^2) \cdot r\, dr\, d\theta$$

$$= 2\pi \left[r^2 - \frac{r^4}{4} \right]_0^{\sqrt{2}}$$

$$= 2\pi [2 - 1]$$

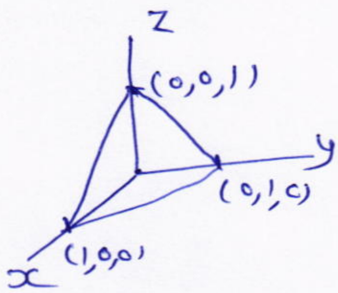
$$= 2\pi$$

Q:7 (14 points) Use **Stokes' theorem** to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} =< x+2z, y-z, x+y >$ and C is the curve of intersection of the plane $x+y+z=1$ with the coordinate planes. (Orient C to be counterclockwise when viewed from above).

Sol:

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2z & y-z & x+y \end{vmatrix}$$

$$= \langle 1+1, -(1-2), 0\rangle = \langle 2, 1, 0\rangle$$



Plane: $z = 1-x-y$

$$g(x,y,z) = x+y+z-1$$

$$\nabla g = \langle 1,1,1\rangle, \ \|\nabla g\| = \sqrt{3}$$

$$\vec{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{\langle 1,1,1\rangle}{\sqrt{3}}$$

$$z_x = -1, \quad z_y = -1, \quad dS = \sqrt{1+1+1}\, dA$$

Stokes' theorem $\displaystyle\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\, dS$

$$= \iint_R \frac{2+1}{\sqrt{3}} \cdot \sqrt{3}\, dA$$

$$= 3\iint_R dA$$

$$= 3\int_{x=0}^{1}\int_{y=0}^{1-x} dy\, dx$$

$$= 3\int_{x=0}^{1} (1-x)\, dx$$

$$= 3\left[x - \frac{x^2}{2}\right]_0^1$$

$$= \frac{3}{2}$$

Q:8 (14 points) Let $\mathbf{F}(x, y, z) = \langle x^3, y^3, z^3 \rangle$. Use the **divergence theorem** to evaluate $\iint_S (\mathbf{F} \cdot \mathbf{n}) dS$, where S is the surface of the region bounded by the sphere $x^2 + y^2 + z^2 = 4$.

Divergence theorem $\iint_S (\vec{F} \cdot \vec{n}) dS = \iiint_D (\nabla \cdot \vec{F}) dV$

$$\nabla \cdot F = 3(x^2 + y^2 + z^2)$$

$$\iiint_D (\nabla \cdot \vec{F}) dV = \iiint_D 3(x^2+y^2+z^2) dV$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_0^2 3\rho^2 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \left[3 \frac{\rho^5}{5} \right]_0^2 \sin\phi \, d\phi \, d\theta$$

$$= 3 \cdot \frac{2^5}{5} \left[-\cos\phi \right]_0^{\pi} \left[\theta \right]_0^{2\pi}$$

$$= 3 \cdot \frac{2^5}{5} \left[-\cos\pi + \cos 0 \right] \cdot 2\pi$$

$$= \frac{12\pi}{5} \cdot 2^5$$