King Fahd University of Petroleum & Minerals
Department of Mathematics
Math 333 Major Exam II
The Second Semester of 2023-2024 (232)

Time Allowed: 120 Minutes

Name: ______________________ ID#: ______________

Section/Instructor: ______________ Serial #: ______________

Key

- Mobiles, calculators and smart devices are not allowed in this exam.
- Write neatly and legibly. You may lose points for messy work.
- Show all your work. No points for answers without justification.

Question #	Marks	Maximum Marks
1		12
2		12
3		12
4		12
5		10
6		14
7		12
8		16
Total		100

Q:1 $(7+5=12$ points) (a) Use the definition of the Laplace transfor to find $\mathcal{L}\left\{\cos(2t)\right\}$.

(b) Evaluate $\mathcal{L}\left\{\sin(t)u(t-\frac{\pi}{2})\right\}$

(a) $\mathcal{L}\{\cos 2t\} = \int_0^\infty e^{-st} \cos 2t \, dt$

$= \cos 2t \frac{e^{-st}}{-s}\Big|_0^\infty - 2\int -\sin 2t \frac{e^{-st}}{-s} dt$

$= \frac{1}{s} - \frac{2}{s}\int_0^\infty e^{-st} \sin 2t \, dt$

$= \frac{1}{s} - \frac{2}{s}\left[\frac{\sin 2t \, e^{-st}}{-s}\Big|_0^\infty - 2\int_0^\infty \cos 2t \frac{e^{-st}}{-s} dt\right.$

$= \frac{1}{s} - \frac{2}{s}\left[0 + \frac{2}{s}\int_0^\infty e^{-st} \cos 2t \, dt\right.$

$= \frac{1}{s} - \frac{4}{s^2}\int_0^\infty e^{-st} \cos 2t \, dt$

$= \frac{1}{s} - \frac{4}{s^2}\mathcal{L}\{\cos 2t\}$

$\left(1+\frac{4}{s^2}\right)\mathcal{L}\{\cos 2t\} = \frac{1}{s}$

$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$

(b) $\mathcal{L}\{\sin t \, u(t-\frac{\pi}{2})\}$

$= \mathcal{L}\{\sin(t-\frac{\pi}{2}+\frac{\pi}{2})u(t-\frac{\pi}{2})\}$

$= \mathcal{L}\{(\sin(t-\frac{\pi}{2})\cos\frac{\pi}{2} + \cos(t-\frac{\pi}{2})\sin\frac{\pi}{2})u(t-\frac{\pi}{2})\}$

$= \mathcal{L}\{\cos(t-\frac{\pi}{2})u(t-\frac{\pi}{2})\}$

$= \frac{s}{s^2+1} e^{-\frac{\pi s}{2}}$

Q:2 (8 + 4 =12 points) Find the following:

(a) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2s-3}\right\}$ (b) $\mathcal{L}\left\{e^{-2t}\cos(5t)\right\}$

(a) $\dfrac{s}{s^2+2s-3} = \dfrac{s}{(s-1)(s+3)}$

$= \dfrac{A}{s-1} + \dfrac{B}{s+3}$

$\Rightarrow s = A(s+3) + B(s-1)$

Let $s=1$. Then $A = \frac{1}{4}$.

Let $s=-3$. Then $B = \frac{3}{4}$

$\therefore \dfrac{s}{s^2+2s-3} = \dfrac{1}{4}\cdot\dfrac{1}{s-1} + \dfrac{3}{4}\cdot\dfrac{1}{s+3}$

$\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2s-3}\right\} = \dfrac{1}{4}\mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\} + \dfrac{3}{4}\mathcal{L}^{-1}\left\{\dfrac{1}{s+3}\right\}$

$= \dfrac{1}{4}e^{t} + \dfrac{3}{4}e^{-3t}$

(b) $\mathcal{L}\{e^{-2t}\cos 5t\}$

$= \left.\dfrac{s}{s^2+5^2}\right|_{s\to s+2}$

$= \dfrac{s+2}{(s+2)^2+5^2}$

Q:3 (12 points) Solve $f(t) = \cos t + \int_0^t e^{-\tau} f(t-\tau)\, d\tau$.

Sol:

$$f(t) = \cos t + e^{-t} * f(t)$$

Taking L.T. on both sides, we get

$$F(s) = \frac{s}{s^2+1} + \frac{1}{s+1} F(s)$$

$$\left(1 - \frac{1}{s+1}\right) F(s) = \frac{s}{s^2+1}$$

$$\left(\frac{s}{s+1}\right) F(s) = \frac{s}{s^2+1}$$

$$F(s) = \frac{s+1}{s^2+1}$$

$$F(s) = \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

Inverting

$$f(t) = \cos t + \sin t$$

Q:4 (12 points) Solve the Initial value problem :

$$y'' + 4y = \delta(t - \frac{\pi}{2}) - \delta(t - \frac{3\pi}{2}), \quad y(0) = 0, \quad y'(0) = 0.$$

$$y'' + 4y = \delta(t - \tfrac{\pi}{2}) - \delta(t - \tfrac{3\pi}{2})$$

Taking Laplace transform on both sides, we get

$$s^2 Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} + 4Y(s) = e^{-\frac{\pi s}{2}} - e^{-\frac{3\pi s}{2}}$$

$$\Rightarrow \quad (s^2 + 4) Y(s) = e^{-\frac{\pi s}{2}} - e^{-\frac{3\pi s}{2}}$$

$$Y(s) = \frac{e^{-\frac{\pi s}{2}}}{s^2+4} - \frac{e^{-\frac{3\pi s}{2}}}{s^2+4}$$

Inverting,

$$y(t) = \frac{1}{2}\sin(2t - \pi)\, u(t - \tfrac{\pi}{2}) - \frac{1}{2}\sin(2t - 3\pi)\, u(t - \tfrac{3\pi}{2})$$

Q:5 (10 points) Let $f_1(x) = x$ and $f_2(x) = x^2$ be orthogonal on $[-2, 2]$. Find constants C_1 and C_2 such that $f_3(x) = x + C_1x^2 + C_2x^3$ is orthogonal to both $f_1(x)$ and $f_2(x)$ on the same interval.

(i) $$(f_1(x), f_3(x)) = \int_a^b f_1(x) f_3(x)\,dx = 0$$

$$\Rightarrow \quad 0 = \int_{-2}^{2} x(x + C_1x^2 + C_2x^3)\,dx$$

$$= \int_{-2}^{2} (x^2 + C_1x^3 + C_2x^4)\,dx$$

$$0 = \frac{x^3}{3} + C_1\frac{x^4}{4} + C_2\frac{x^5}{5}\Big|_{-2}^{2}$$

$$= \frac{16}{3} + \frac{64}{5}C_2$$

$$\Rightarrow \boxed{C_2 = -\frac{5}{12}}$$

(ii) $$(f_2(x)\, f_3(x)) = \int_{-2}^{2} x^2(x + C_1x^2 + C_2x^3)\,dx = 0$$

$$\Rightarrow \quad \frac{x^4}{4} + C_1\frac{x^5}{5} + C_2\frac{x^6}{6}\Big|_{-2}^{2} = 0$$

$$\Rightarrow \quad \frac{64}{5}C_1 = 0$$

$$\Rightarrow \boxed{C_1 = 0}$$

Q:6 ($10+4=14$ points) (a) Find the Fourier series of the function $f(x)=\begin{cases} 0, & -\pi<x<0 \\ x^2, & 0\le x<\pi. \end{cases}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{1}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x^2 \cos nx\,dx = \frac{1}{\pi}\left[\frac{x^2 \sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{2x}{n}\sin nx\,dx\right]$$

$$= -\frac{2}{n\pi}\int_0^{\pi} x\sin nx\,dx = -\frac{2}{n\pi}\left[-\frac{x}{n}\cos nx\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}\,dx\right]$$

$$= \frac{2\pi\cos n\pi}{n^2\pi} + \frac{1}{n^2}\left(\sin nx\right)_0^{\pi}$$

$$= \frac{2}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x^2 \sin nx\,dx = \frac{1}{\pi}\left[-\frac{x^2}{n}\cos nx\Big|_0^{\pi} + \int_0^{\pi}\frac{2x}{n}\cos nx\,dx\right]$$

$$= -\frac{\pi}{n}(-1)^n + \frac{2}{n\pi}\left[\frac{x}{n}\sin nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right]$$

$$= -\frac{\pi}{n}(-1)^n + \frac{2}{n^2\pi}\cdot\frac{1}{n}\cos nx\Big|_0^{\pi}$$

$$= -\frac{\pi}{n}(-1)^n + \frac{2}{n^3\pi}\left(\cos n\pi - 1\right)$$

$$= (-1)^{n+1}\frac{\pi}{n} + \frac{2}{n^3\pi}\left((-1)^n - 1\right)$$

$$\therefore\quad f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{n^2}\cos nx + \left(\frac{\pi}{n}(-1)^{n+1} + \frac{2}{n^3\pi}\left((-1)^n-1\right)\right)\sin nx\right.$$

(b) Use the results of (a) to show that

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\dots$$

At $x=0$, the series converges to zero

$$0 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}\cos 0 + 0$$

$$= \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}$$

$$= \frac{\pi^2}{6} + 2\left(-1 + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right)$$

$$\frac{\pi^2}{16} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

Q:7 (12 points) Expand $f(x) = x, \quad -\pi < x < \pi$ in a Fourier sine series.

Since $f(x)$ is an odd function, therefore $a_0 = 0$, $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left[\frac{x \cos nx}{-n}\Big|_0^{\pi} + \int_0^{\pi} 1 . \frac{\cos nx}{n} dx \right]$$

$$= \frac{2\pi}{-n\pi}(-1)^n + \frac{2}{n\pi}\int_0^{1} \cos nx \, dx$$

$$= \frac{2}{n}(-1)^{n+1} + \frac{2}{n\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}$$

$$= \frac{2}{n}(-1)^{n+1} + 0$$

Thus

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx .$$

Q:8 (16 points) (a) Find the eigenvalues and the eigenfunctions of the **Sturm-Liouville problem:**

$$x^2y'' + xy' + \lambda y = 0, \quad y(1) = 0, \quad y(5) = 0.$$

(b) Put the differential equation in self-adjoint form.
(c) Give an orthogonality relation.

Sol: Auxiliary equation $m(m-1)+m+\lambda=0$
$\Rightarrow m = \pm i\sqrt{\lambda}$

(i) $\underline{\lambda=0}$: $m=0,0$.
General Solution $y = c_1 x^0 + c_2 x^0 \ln x$
$y(1)=0 \Rightarrow c_1=0$; $y(5)=0=c_2 \ln 5 \Rightarrow c_2=0$
$y=0$ trivial solution

(ii) $\underline{\lambda>0}$: $y = c_1 \cos(\sqrt{\lambda}\ln x) + c_2 \sin(\sqrt{\lambda}\ln x)$
$y(1)=0 \Rightarrow c_1=0$
$y(5)=0 \Rightarrow 0 = c_2 \sin(\sqrt{\lambda}\ln 5)$
$\Rightarrow c_2 \sin(\sqrt{\lambda}\ln 5) = \sin n\pi, \; n=1,2,3,\ldots$
$\Rightarrow \sqrt{\lambda}\ln 5 = n\pi \quad (c_2 \neq 0)$
$\Rightarrow \lambda = \dfrac{n^2\pi^2}{(\ln 5)^2}, \; n=1,2,3,\ldots$

Eigenfunctions are
$$y = c\sin\left(\frac{n\pi}{\ln 5}\ln x\right)$$

(iii) $\underline{\lambda<0}$: $m=\pm\sqrt{\lambda}$ G.S. $y = c_1 x^{\sqrt{\lambda}} + c_2 x^{-\sqrt{\lambda}}$
$y(1)=0 \Rightarrow c_1 x + \dfrac{c_2}{x} = 0$
$y(5)=0 \Rightarrow c_1 x^{\sqrt{5}} + \dfrac{c_2}{x^{\sqrt{5}}} = 0$ $\Rightarrow c_1 = 0 = c_2$
$\Rightarrow y=0$ trivial solution

(b) DE can be written as
$$y'' + \frac{1}{x}y' + \frac{\lambda}{x^2}y = 0 \quad -①$$
I.F. $e^{\int \frac{1}{x}dx} = e^{\ln x} = x$
Multiplying ① by x, we get
$$xy'' + y' + \frac{\lambda}{x}y = 0$$
or
$$\frac{d}{dx}(xy') + \frac{\lambda}{x}y = 0$$

(c) Orthogonality relation $= \int_a^b p(x)\, y_n(x)\, y_m(x)\, dx = 0$
$$\int_1^5 \frac{1}{x}\sin\left(\frac{n\pi}{\ln 5}\ln x\right)\sin\left(\frac{m\pi}{\ln 5}\ln x\right)dx = 0$$