King Fahd University of Petroleum & Minerals
Department of Mathematics
Math 333 Final Exam
The Second Semester of 2023-2024 (232)

Time Allowed: 160 Minutes

Name: ______________________ ID#: ______________

Section/Instructor: ______________ Serial #: ______________

- Mobiles, calculators and smart devices are not allowed in this exam.
- Write neatly and legibly. You may lose points for messy work.
- **Show all your work. No points for answers without justification.**

Question #	Marks	Maximum Marks
1		13
2		15
3		18
4		15
5		18
6		18
7		18
8		10
9		15
Total		140

Q:1 (13 points) Let $\mathbf{F}(x,y,z) = \langle xy, y^2z, z^3\rangle$. Use the **divergence theorem** to evaluate $\iint_S (\mathbf{F}\cdot\mathbf{n})dS$, where S is the unit cube defined by $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.

(**Divergence Theorem:** $\iint_S (\mathbf{F}\cdot\mathbf{n})dS = \iiint_D (\nabla\cdot\mathbf{F})dV$)

$$\nabla\cdot\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle xy, y^2z, z^3\rangle$$

$$= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2z) + \frac{\partial}{\partial z}(z^3)$$

$$= y + 2yz + 3z^2$$

$$\iint_S (\vec{F}\cdot\vec{n})dS = \iiint_D (\nabla\cdot F)\,dV$$

$$= \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} (y + 2yz + 3z^2)\,dx\,dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=0}^{1} \left[xy + 2xyz + 3xz^2\right]_0^1 dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=0}^{1} \left[y + 2yz + 3z^2\right] dy\,dz$$

$$= \int_{z=0}^{1} \left[\frac{y^2}{2} + y^2z + 3yz^2\right]_0^1 dz$$

$$= \int_{z=0}^{1} \left[\frac{1}{2} + z + 3z^2\right] dz$$

$$= \frac{1}{2}z + \frac{z^2}{2} + z^3\Big|_0^1$$

$$= \frac{1}{2} + \frac{1}{2} + 1 = 2.$$

9.16

Example 2, Page 562

Q:2 (15 points) Find the eigenvalues and the eigenfunctions of the **Sturm-Liouville problem:**

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y\prime(L) = 0.$$

(i) Let $\lambda = 0$. Then $y'' = 0 \Rightarrow y = c_1 x + c_2$

$y' = c_1$

$y'(0) = 0 \Rightarrow c_1 = 0$

$\therefore \boxed{y = c_2}$

(ii) $\lambda = -\alpha^2 < 0$; $y'' - \alpha^2 y = 0$

$y = c_3 \cosh \alpha x + c_4 \sinh \alpha x$

$y' = c_3 \alpha \sinh \alpha x + \alpha c_4 \cosh \alpha x$

$y'(0) = 0 \Rightarrow c_4 = 0$

$y'(L) = 0 \Rightarrow c_3 = 0$

$\Rightarrow y = 0$ trivial Solution

(iii) $\lambda = \alpha^2 > 0$: $y'' + \alpha^2 y = 0$

$y = c_5 \cos \alpha x + c_6 \sin \alpha x$

$y' = -\alpha c_5 \sin \alpha x + \alpha c_6 \cos \alpha x$

$y'(0) = 0 \Rightarrow c_6 = 0$

$y'(L) = 0 \Rightarrow c_5 \sin \alpha L = 0$

$\sin \alpha L = 0 \quad (c_5 \neq 0)$

$= \sin n\pi$

$\alpha L = n\pi \Rightarrow \alpha = \frac{n\pi}{L}$

$\lambda = \frac{n^2 \pi^2}{L^2}$

$y = c \cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, 3, \dots$

$\frac{12.5}{\text{Exercise } 3}$

Q:3 (18 points) Use separation of variables method to solve the problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \ 0 < x < \pi, \ t > 0,$$

subject to the boundary and initial conditions

$$u(0,t) = u(\pi,t) = 0, t > 0$$

$$u(x,0) = x, \ 0 < x < \pi.$$

Sol:

Let $u(x,t) = X(x)T(t) \Rightarrow X''T = XT' \Rightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda$

$u(0,t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0$

$u(\pi,t) = 0 \Rightarrow X(\pi)T(t) = 0 \Rightarrow X(\pi) = 0$

(i) $\frac{T'}{T} = -\lambda \Rightarrow T' + \lambda T = 0 \Rightarrow T(t) = c\,e^{-\lambda t}$

(ii) $X'' + \lambda X = 0$, $X(0) = 0$, $X(\pi) = 0$.

(a) $\lambda = 0$: $X'' = 0 \Rightarrow X(x) = c_1 + c_2 x$

$X(0) = 0 \Rightarrow c_1 = 0$

$X(\pi) = 0 \Rightarrow c_2 = 0$ $\Rightarrow X(x) = 0$ trivial solution

(b) $\lambda = -\alpha^2\ (\alpha \neq 0)$: $X'' - \alpha^2 X = 0 \Rightarrow X = c_3 \cosh\alpha x + c_4 \sinh\alpha x$

$X(0) = 0 \Rightarrow c_1 = 0$

$X(\pi) = 0 \Rightarrow c_2 = 0$ $X(x) = 0$ trivial solution

(c) $\underline{\lambda = \alpha^2\ (\alpha \neq 0)}$: $X'' + \alpha^2 X = 0 \Rightarrow X(x) = c_5 \cos\alpha x + c_6 \sin\alpha x$

$X(0) = 0 \Rightarrow c_5 = 0$

$X(\pi) = 0 \Rightarrow c_6 \sin\alpha\pi = 0$

$\Rightarrow \sin\alpha\pi = \sin n\pi \quad (c_6 \neq 0)$

$\Rightarrow \alpha\pi = n\pi \Rightarrow \alpha = n,\ n = 1, 2, 3, \ldots$

$X_n(x) = c_6 \sin nx$ and $T(t) = c\,e^{-n^2 t}$

Product solution $u_n(x,t) = A_n e^{-n^2 t} \sin nx,\ n = 1, 2, 3, \ldots$

By superposition principle, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 t} \sin nx$$

Using $u(x,0) = x \Rightarrow x = \sum_{n=1}^{\infty} A_n \sin nx$ (Sine Series)

$$A_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx = \frac{2}{\pi}\left[-\frac{x}{n}\cos nx\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\, dx\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{n}\cos n\pi + 0 + 0\right] = -\frac{2}{n}(-1)^n$$

Hence

$$u(x,t) = \sum_{n=1}^{\infty}\left(-\frac{2}{n}\right)(-1)^n e^{-n^2 t} \sin nx.$$

13.3

Similar Exercise 3

Q:4 (15 points) Consider the Laplce equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \ 0 < x < \pi, \ 0 < y < \pi,$$

subject to the boundary and initial conditions

$$u(0, y) = 0, \ u(\pi, y) = 0, \quad 0 < y < \pi,$$

$$u(x, 0) = 0, \ u(x, \pi) = 1, \quad 0 < x < \pi.$$

If $X(x) = B_n \sin(\mathrm{nx})$ for $\lambda = n^2$ is the nontrivial solution obtained by using separation of variables with $u(x, y) = X(x)Y(y)$, find the general solution of the problem.

Let $u(x,y) = X(x)Y(y)$, we get

$X'' + \lambda X = 0 \qquad Y'' - \lambda Y = 0$

$X(0)=0, X(\pi)=0 \qquad Y(0)=0$

Given solution $X(x) = B_n \sin nx$, $\lambda = n^2$

$Y'' - \lambda Y = 0 \Rightarrow Y(y) = C \cosh ny + D \sinh ny$

$Y(0) = 0 \Rightarrow C = 0$.

The general solution is

$u(x,y) = \sum_{n=1}^{\infty} A_n \sin nx \sinh ny \quad ; A_n = B_n D$

$u(x,\pi) = 1 \Rightarrow 1 = \sum_{n=1}^{\infty} B_n \sinh n\pi \sin nx$

$B_n \sinh n\pi = \frac{2}{\pi} \int_0^1 \sin nx \, dx$

$= -\frac{2}{\pi n} \cos nx \Big|_0^\pi = \frac{2}{n\pi}[1-(-1)^n]$

$B_n = \frac{2}{n\pi \sinh n\pi}[1-(-1)^n] =$

$\therefore u(x,y) = \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]}{n\pi \sinh n\pi} \sinh ny \sin nx$

13.5

Similar Exercise 1

Q:5 (18 points) Use Laplace transform to solve the problem

$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \; x>0, \; t>0,$$

subject to the boundary and initial conditions

$$u(0,t) = f(t), \quad \lim_{x\to\infty} u(x,t) = 0, \; t>0,$$

$$u(x,0) = 0, \; \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \; x>0.$$

Sol: $\mathcal{L}\{4u_{xx}\} = \mathcal{L}\{u_{tt}\}$, where $\mathcal{L}\{u(x,t)\} = U(x,s)$

$$4\frac{d^2U}{dx^2} = s^2 U(x,s) - s u(x,0) - u_t(x,0)$$

$$= s^2 U - 0 - 0$$

$$\frac{d^2U}{dx^2} - \frac{s^2}{4}U = 0$$

General Solution: $U(x,s) = c_1 e^{-\frac{s}{2}x} + c_2 e^{\frac{s}{2}x}$ — ⊗

$$\boxed{\begin{array}{l}\mathcal{L}\{\lim_{x\to\infty} u(x,t)\} = 0 \Rightarrow \lim_{x\to\infty} U(x,s) = 0 \\ \mathcal{L}\{u(0,t)\} = \mathcal{L}\{f(t)\} \Rightarrow U(0,s) = F(s)\end{array}}$$

Using $\lim_{x\to\infty} U(x,s) = 0$ in ⊗, we get $c_2 = 0$

$$U(x,s) = c_1 e^{-\frac{s}{2}x}$$

Using $U(0,s) = F(s) \Rightarrow c_1 = F(s)$

$$\therefore U(x,s) = F(s)\, e^{-\frac{s}{2}x}$$

Inverting

$$u(x,t) = f\left(t - \frac{x}{2}\right) \mathcal{U}\left(t - \frac{x}{2}\right)$$

15.2
Exercise 3

Q:6 (18 points) Use separation of variables to solve the problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}, \; 0 < r < C, \; t > 0,$$

subject to the boundary conditions

$$u(C,t) = 0, \; t > 0,$$
$$u(r,0) = 0, \; 0 < r < C,$$
$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 100, \;\; 0 < r < C.$$

Solution $u(r,t)$ is bounded at $r = 0$.

Let $u(r,t) = R(r)\,T(t)$. Then $\dfrac{R'' + \frac{1}{r}R'}{R} = \dfrac{T''}{T} = -\lambda$

$\Rightarrow \; rR'' + R' + \lambda r R = 0, \; T'' + \lambda T = 0$

$u(C,t) = 0 \Rightarrow R(C) = 0, \quad u(r,0) = 0 \Rightarrow T(0) = 0$

Let $\lambda = \alpha^2$. Then $rR'' + R' + \alpha^2 r R = 0$

$\Rightarrow R(r) = c_1 J_0(\alpha r)$

and $R(C) = 0 \Rightarrow J_0(\alpha C) = 0$.

α_i's are the non-zero values such that $J_0(C\alpha_i) = 0 \Rightarrow R = J_0(\alpha_i r)$

For $\lambda = \alpha_i^2$: $\quad T = c_2 \cos\alpha_i t + c_3 \sin\alpha_i t$

$T(0) = 0 \Rightarrow c_2 = 0$ and $T = c_3 \sin(\alpha_i t)$

Product solution is

$$u(r,t) = \sum_{i=1}^{\infty} A_i \sin(\alpha_i t)\, J_0(\alpha_i r)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 100 \Rightarrow 100 = \sum_{i=1}^{\infty} A_i \alpha_i \, J_0(\alpha_i r)$$

$$\Rightarrow A_i \alpha_i = \frac{2}{C^2 J_1^2(\alpha_i C)} \int_0^C 100 \cdot r \cdot J_0(\alpha_i r)\, dr$$

Let $\alpha_i r = x$. Then

$$A_i \alpha_i = \frac{200}{C^2 \alpha_i^2 J_1^2(\alpha_i C)} \int_0^{C\alpha_i} x J_0(x)\, dx$$

$$A_i = \frac{200}{C^2 \alpha_i^3 J_1^2(\alpha_i C)} \int_0^{C\alpha_i} \frac{d}{dx}\left[x J_1(x)\right] dx$$

$$= \frac{200}{C^2 \alpha_i^3 J_1^2(\alpha_i C)} \left[x J_1(x)\right]_0^{C\alpha_i}$$

$$= \frac{200}{C \alpha_i^2 J_1(\alpha_i C)}$$

Hence

$$u(r,t) = \frac{200}{C} \sum_{i=1}^{\infty} \frac{J_0(\alpha_i r)}{\alpha_i^2 J_1(\alpha_i C)} \sin(\alpha_i t).$$

14.2
Example 1

Q:7 (18 points) Find the steady-state temperature $u(r,\theta)$ in a sphere of radius C by solving the problem

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{cot\theta}{r^2}\frac{\partial u}{\partial \theta}=0,\ 0<r<c,\ 0<\theta<\pi,$$

$$u(C,\theta)=\ 5.$$

Let $u(r,\theta)=R(r)\,\Theta(\theta)$. Then

$$R''\Theta+\frac{2}{r}R'\Theta+\frac{1}{r^2}R\Theta''+\frac{\cot\theta}{r^2}R\Theta'=0$$

$$\Leftrightarrow \frac{r^2R''+2rR'}{R}=-\frac{\Theta''+\cot\theta\,\Theta'}{\Theta}=\lambda$$

$$\Rightarrow \Theta''+\cot\theta\,\Theta'+\lambda\Theta=0 \text{ and } r^2R''+2rR'-\lambda R=0$$

Let $x=\cos\theta$. Then

$$(1-x^2)\,\Theta''(x)-2x\,\Theta'(x)+\lambda\Theta(x)=0$$

For $\lambda=n(n+1)$, $n=0,1,2,\dots$, $\Theta_n(x)=P_n(x)$; $\Theta_n(x)=P_n(\cos\theta)$

Now $r^2R''+2rR'-\lambda R=0$

$$\Rightarrow R=c_1r^n+c_2r^{-(n+1)}$$

$u(r,\theta)$ is bounded $\Rightarrow c_2=0$.

Product Solution:

$$u(r,\theta)=\sum_{n=0}^{\infty}A_n r^n P_n(\cos\theta)$$

Using $u(c,\theta)=5$, we get

$$A_nc^n=\frac{2n+1}{2}\int_0^{\pi}5\,P_n(\cos\theta)\sin\theta\,d\theta$$

By orthogonality $A_n=0,\ n\neq0$

$$A_0=\frac{5}{2}\int_0^{\pi}P_0(\cos\theta)P_0(\cos\theta)\sin\theta\,d\theta$$

$$=\frac{5}{2}\int_{-1}^{1}P_0(x)P_0(x)\,dx$$

$$=\frac{5}{2}\int_{-1}^{1}dx=\frac{5}{2}\cdot2=5$$

Hence

$$u(r,\theta)=A_0P_0(\cos\theta)$$
$$=5\cdot1$$
$$=5$$

14.3

Excercise 3

Q:8 (10 points) Find the Fourier integral representation of the piecewise-continuous function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x < 2. \\ 0, & x > 2. \end{cases}$$

$$A(\alpha) = \int_{-\infty}^{\infty} f(x) \cos \alpha x \, dx$$

$$= \int_{-\infty}^{0} 0 \cdot \cos \alpha x \, dx + \int_{0}^{2} 1 \cdot \cos \alpha x \, dx + \int_{2}^{\infty} 0 \cdot \cos \alpha x \, dx$$

$$= \int_{0}^{2} \cos \alpha x \, dx = \frac{\sin 2\alpha}{\alpha}$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x) \sin \alpha x \, dx$$

$$= \int_{0}^{2} 1 \cdot \sin \alpha x \, dx$$

$$= -\frac{\cos \alpha x}{\alpha} \Big|_{0}^{2}$$

$$= \frac{1 - \cos 2\alpha}{\alpha}$$

Fourier integral representation is

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \left[\left(\frac{\sin 2\alpha}{\alpha} \right) \cos \alpha x + \left(\frac{1 - \cos 2\alpha}{\alpha} \right) \sin \alpha x \right] d\alpha$$

15.3
Example 1

Q:9 (15 points) Use Fourier sine transform to solve

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},\ x > 0,\ t > 0,$$

subject to the conditions

$$u(0,t) = 2,\quad t > 0,$$
$$u(x,0) = 0,\quad x > 0.$$

Sol:
$$\mathcal{F}_s\{k\, u_{xx}\} = \mathcal{F}_s\{u_t\}$$

$$-\alpha^2 k\, U(\alpha,t) + \alpha k\, u(0,t) = \frac{dU}{dt}$$

$$\frac{dU}{dt} + \alpha^2 k U = 2\alpha k$$

$$U_c(\alpha,t) = C e^{-\alpha^2 k t}$$

Let $U_p = A$. Then $A = \frac{2\alpha k}{\alpha^2 k} = \frac{2}{\alpha}$

General Solution:
$$U(\alpha,t) = C e^{-\alpha^2 k t} + \frac{2}{\alpha} \quad \text{———} (*)$$

Now $\mathcal{F}_s\{u(x,0)\} = 0 \Rightarrow U(\alpha,0) = 0 \quad \text{—} (**)$

Using $(**)$ in $(*)$, we get $C = -\frac{2}{\alpha}$

Thus
$$U(\alpha,t) = -\frac{2}{\alpha} e^{-\alpha^2 k t} + \frac{2}{\alpha}$$

Inverting
$$u(x,t) = \frac{2}{\pi}\int_0^\infty \left(-\frac{2}{\alpha} e^{-\alpha^2 k t} + \frac{2}{\alpha}\right) \sin\alpha x \, d\alpha$$

15.4
Similar Excercise 3