King Fahd University of Petroleum & Minerals
Department of Mathematics
Math 333: Methods of Applied Mathematics I
Second Major Exam, Semester 241
Section 01, Instructor: Dr. Rajai S. Alassar

November 18, 2024 at 6:00 pm Time Allowed: 90 minutes

Name: ____ I.D. # ____________________

Question No.	Maximum Marks	Marks
1	4+3+3+4 = 14	
2	5	
3	7	
4	5	
5	7	
6	4	
7	8	
Total	50	

2. **Laplace Transform formulas:**

$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$, $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

$\mathcal{L}\{f(t)\, e^{at}\} = F(s-a)$, $\mathcal{L}\{t^n\, f(t)\} = (-1)^n \dfrac{d^n}{ds^n}F(s)$, $\mathcal{L}\{\delta(t-a)\} = e^{-as}$

$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$, and $\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$

$f(t) * g(t) = \displaystyle\int_0^t f(\tau)\, g(t-\tau)\, d\tau$, $\mathcal{L}\{f(t) * g(t)\} = F(s)\, G(s)$

3. **Fourier Sine and Cosine Series on** $(0, L)$**:**

Fourier Series on $(-p, p)$ $f(x) = \dfrac{a_o}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x$,

$a_o = \dfrac{1}{p}\displaystyle\int_{-p}^{p} f(x)\, dx$, $a_n = \dfrac{1}{p}\displaystyle\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\, dx$, $b_n = \dfrac{1}{p}\displaystyle\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\, dx$

Half Range Fourier Cosine Series On $(0, L)$, $f(x) = \dfrac{a_o}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$,

where $a_o = \dfrac{2}{L}\displaystyle\int_0^L f(x)\, dx$, $a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{n\pi}{L}x\, dx$,

Half Range Fourier Sine Series On $(0, L)$, $f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$,

where $b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{n\pi}{L}x\, dx$

1. Find the following.

   a) $\mathcal{L}\{f(t)\}$ using the definition, where $f(t) = \begin{cases} 2-t & if \quad 0 < t < 2 \\ 0 & if \quad t \geq 2 \end{cases}$.

   b) $\mathcal{L}\{(t+1)\,u(t-1)\}$, where u is the unit-step function.

c) $\mathcal{L}\{t^2\ \sin t\}$

d) $\mathcal{L}^{-1}\{\frac{s+5}{s^2+6s+34}\}$

2. Use Laplace Transform to solve the initial value problem

$$\frac{dy}{dt} + 6\,y + 9\int_0^t y(\tau)\,d\tau = 1, \quad y(0) = 0$$

3. Use Laplace Transform to solve the initial value problem

$$\frac{d^2y}{dt^2} + y = \delta(t - 2\pi), \quad y(0) = 0, y'(0) = 1$$

Sketch the solution on the interval $[\,0\,, 4\pi\,]$.

4. Show that $f_1(x) = 1$ and $f_2(x) = 1 - x$ are orthogonal on $[0, \infty)$ with respect to the weight function $w(x) = e^{-x}$. Find the norm of $f_1(x)$.

5. Find the Fourier series of $f(x) = \begin{cases} 0 & -1 < x < 0 \\ x & 0 \leq x < 1 \end{cases}$. You may need the formulas

$$\int x\,cos(\,\alpha x)dx = \frac{cos(\alpha x)}{\alpha^2} + \frac{x\,sin(\alpha x)}{\alpha} + c \quad , \quad \int x\,sin(\,\alpha x)dx = \frac{sin(\alpha x)}{\alpha^2} - \frac{x\,cos(\alpha x)}{\alpha} + c$$

6. The Fourier series of $f(x) = \begin{cases} \pi^2 & -\pi < x < 0 \\ \pi^2 - x^2 & 0 \le x < \pi \end{cases}$ is given by

$$f(x) = \frac{5\pi^2}{6} + \sum_{n=1}^{\infty} \left(\frac{2\,(-1)^{n+1}}{n^2} \cos(nx) + \left(\frac{\pi}{n}\,(-1)^n + \frac{2(1-(-1)^n)}{n^3\,\pi} \right) \sin(n\,x) \right)$$

Use the given Fourier series to evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Explain your answer.

7. Consider $y'' + \lambda y = 0, \; y(0) + y'(0) = 0, \; y(1) = 0$.

It can be shown that the boundary value problem possesses a trivial solution for $\lambda < 0$. Find the eigenfunctions and the equation that defines the eigenvalues for the given boundary value problem.