King Fahd University of Petroleum & Minerals
Department of Mathematical Sciences

Math 333 – Final Exam

Semester I, 2024/2025 (241) Date: December 22, 2024 Time:7:00 pm-9:30 pm

Name: ______________________________ I.D. # ____________________

QUESTION	GRADE
1	/6
2	/5
3	/7
4	/8
5	/6
6	/3
TOTAL	/35

1. The temperature in a semi-infinite rod (with unit thermal diffusivity) is determined from

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, x > 0, t > 0$$

$$u(0,t) = 1, \quad t > 0 \qquad\qquad u(x,0) = 0, \quad x > 0$$

Use a **proper** Fourier transform to find the temperature $u(x,t)$ in the rod. Leave your answer in terms of integrals.

2. a) Find the Fourier integral representation of $f(x) = \begin{cases} 1 & if \ -1 < x < 1 \\ 0 & otherwise \end{cases}$.

b) Evaluate the integral $\int_0^\infty \frac{\sin w}{w}\, dw$.

3. Use Laplace transform to solve

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad x > 0, \qquad t > 0$$

$$u(0, t) = 2, \qquad \lim_{x \to \infty} u_x(x, t) = 0, \qquad t > 0$$

$$u(x, 0) = 1, \qquad u_t(x, 0) = 0, \qquad x > 0$$

Based on your solution, plot a graph for $u(x, 3)$.

4. Steady-state temperature in cylindrical coordinates is given by $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}=0.$
To find the temperature distribution in a cylinder defined by $0 \le r \le 1,\ \ 0 \le z \le 1$, we se separation of variables (i.e. $u(z,r)=R(r)\,Z(z)$). As a result of the separation, one of the resulting differential equations can be written as $\frac{Z''}{Z}=\lambda$. Considering the radial symmetry of the problem (no variation in the θ-direction) and if the boundary conditions are given by
$u(1,z)=0,\ 0<z<1$ $\qquad u_z(r,0)=0,\ 0<r<1$ $\qquad u(r,1)=100,\ 0<r<1$

a) Show that $\lambda=0$ is not an eigenvalue.
b) Consider only $\lambda=\alpha^2;\ \alpha>0$ (i.e. $\frac{Z''}{Z}=\alpha^2$), solve the problem completely. (Clearly write down the final series solution and the equation that defines the eigenvalues).

5. Consider the vector field $\vec{F}(x, y, z) = 5y\, i - 5x\, j + 3\, k$. Assume that the upward-oriented surface S is that portion of the plane $z = 1$ within the cylinder $x^2 + y^2 = 4$. Verify Stokes' theorem.

6. If the Fourier-Legendre expansion of $f(x) = e^x,\ -1 < x < 1$ is $\sum_{n=0}^{\infty} c_n\, P_n(x)$. Calculate c_2 .