King Fahd University of Petroleum and Mineral
College of Computing and Mathematics
Department of Mathematics

MATH341 – Advanced Calculus I

Academic Year 2022-23

Term 221

Major Exam 1

Name:	Solution
ID#:	

- ***The answers must be fully supported by logical arguments to get full credit***

Question	Score	Max Score
1		17
2		20
3		24
4		22
5		17
Total		**100**

Time allowed: **100** Minutes

Exercise 1

Fill in the blank with the most appropriate term/expression.

(a) If $a \in \mathbb{R}$ is such that $0 \le a < \varepsilon$ for every $\varepsilon > 0$, then $a = 0$

(b) **The completeness property of** $\mathbb{R}$. Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.

(c) **The Density Theorem.** If x and y are any real numbers with $x < y$, then there exists a rational number r such that $x < r < y$

(d) **Archimedean Property**. If $x \in \mathbb{R}$, then there exists $n \in \mathbb{N}$ such that $x \le n$

(e) **Monotone Convergence Theorem**. A monotone sequence of real numbers is convergent if and only if it is bounded. If (x_n) is a bounded decreasing sequence, then

$$\lim(x_n) = \inf\{x_n : n \in \mathbb{N}\}$$

(f) A sequence (x_n) of real numbers is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there exists $H(\varepsilon) \in \mathbb{N}$ such that for all $m, n \geq H(\varepsilon)$ we have

$|x_n - x_m| < \varepsilon$, $m, n \in \mathbb{N}$

(g) **Monotone Subsequence Theorem**. If (x_n) is a sequence of real numbers, then there is a subsequence of (x_n) that is monotone

Exercise 2

(a) If $c > 1$, show that $c^n \geq c$ for all $n \in \mathbb{N}$, and that $c^n > c$ for $n > 1$.

(b) Find all $x \in \mathbb{R}$ that satisfy $|2x - 4| + |x + 2| < 7$

(a) proof. If $c > 1$, then we can write $c = 1 + a$, $a > 0$.

We have $c^n = (1+a)^n \geq 1 + na \geq 1 + a = c$ if $n \geq 1$

and $c^n = (1+a)^n \geq 1 + na > 1 + a = c$ if $n > 1$

(b) If $x \geq 2$, then we have

$$2x - 4 + x + 2 < 7$$

$$\Rightarrow 3x < 9 \Rightarrow x < 3$$

Hence, $2 \leq x < 3$

If $-2 \leq x < 2$, then we have

$$-2x + 4 + x + 2 < 7$$

$$\Rightarrow -x < 1 \Rightarrow x > -1$$

Thus, $-1 < x < 2$

If $x < -2$, then we have

$$-2x + 4 - x - 2 < 7$$

$$\Rightarrow -3x < 5 \Rightarrow x > -\frac{5}{3}$$

$$\{x < -2\} \cup \{x > -5/3\} = \phi$$

We obtain the solution: $-1 < x < 3$

Exercise 3 Let S be a nonempty bounded set in $\mathbb{R}$.

(a) Let $a > 0$ and $aS = \{as: s \in S\}$. Show that

$$\inf(aS) = a \inf S \quad \text{and} \quad \sup(aS) = a \sup S$$

(b) Let $b < 0$ and $bS = \{bs: s \in S\}$. Show that

$$\inf(bS) = b \sup S \quad \text{and} \quad \sup(bS) = b \inf S$$

Proof.

(a) Let $u = \inf(S)$. We have $s \geq u \quad \forall s \in S$.

If $a > 0$, then $as \geq au \quad \forall s \in S$.

Hence, au is a lower bound of aS. Let v be a lower bound of aS. Then $v \leq as \quad \forall s \in S$.

Since $a > 0$, $\frac{v}{a} \leq s \quad \forall s \in S$. Thus, $\frac{v}{a}$ is a lower bound of S. We have $\frac{v}{a} \geq \inf(S) = u$.

Therefore, $v \geq au$. So, $\inf(aS) = au = a\inf(S)$

Similarly, we can show that $\sup(aS) = a\sup(S)$.

(b) If $w = \sup(S)$, $s \leq w \quad \forall s \in S$. If $b < 0$,

we have $bs \geq bw \quad \forall s \in S$. So, bw is a lower bound of bS. Let v be a lower bound of bS. Thus, $v \leq bs \quad \forall s \in S$. Since $b < 0$,

$\frac{v}{b} \geq s \quad \forall s \in S$. We have that $\frac{v}{b}$ is an upper bound of S. Hence, $\frac{v}{b} \geq \sup(S) = w$. Since $b < 0$,

$v \leq bw$.

Therefore, $\inf(bS) = bw = b\sup(S)$

If $u = \inf(S)$, $s \geq u$ $\forall s \in S$. If $b < 0$, we have $bs \leq bu$ $\forall s \in S$. so, bu is an upper bound of bS. Let z be an upper bound of bS. Thus, $z \geq bs$ $\forall s \in S$. Since $b < 0$, $\frac{z}{b} \leq s$ $\forall s \in S$. we have that $\frac{z}{b}$ is a lower bound of S. Hence, $\frac{z}{b} \leq \inf(S) = u$. since $b < 0$,

$$z \geq bu.$$

Therefore, $\sup(bS) = bu = b\inf(S)$

Exercise 4 Let $x_1 \geq 3$ and $x_{n+1} = 2 + \sqrt{x_n - 2}$ for $n \in \mathbb{N}$.

(a) Prove that (x_n) is monotone

(b) Show that (x_n) is bounded

(c) Is the sequence convergent? and why? If yes, find its limit.

(a) proof. By induction we show that $x_{n+1} \leq x_n$, $\forall n \in \mathbb{N}$.
Since $x_1 \geq 3$, $x_1 - 2 \geq 1$. We have

$$x_1 - 2 \geq \sqrt{x_1 - 2} \Rightarrow x_1 \geq 2 + \sqrt{x_1 - 2} = x_2$$

So, it is true for $n=1$. Assume that it is true for $n=k$: $x_{k+1} \leq x_k$. We obtain

$$x_{k+2} = 2 + \sqrt{x_{k+1} - 2} \geq 2 + \sqrt{x_k - 2} = x_{k+1}$$

So, it is true for $n=k+1$. Hence, the sequence is decresing (monotone)

(b) $x_1 \geq 3$. Assume that $x_k \geq 3$. We will show that $x_{k+1} \geq 3$. $x_{k+1} = 2 + \sqrt{x_k - 2} \geq 2 + \sqrt{3-2} = 3$.

So, $x_n \geq 3$ $\forall n \in \mathbb{N}$ by induction.
Since (x_n) is bounded, we have $3 \leq x_n \leq x_1$ for all $n \in \mathbb{N}$. Hence, (x_n) is bounded.

(c) The sequence is decreasing and bounded, the sequence is convergent. Let $x = \lim(x_n) = \lim(x_{n+1})$

$$\lim(x_{n+1}) = 2 + \sqrt{\lim(x_n) - 2} \Rightarrow x = 2 + \sqrt{x-2}$$

$$(x-2)^2 = x - 2 \Rightarrow x - 2 = 0 \text{ or } x - 2 = 1$$

$$\Rightarrow x = 2 \qquad \Rightarrow x = 3$$

Since $x_n \geq 3$ $\forall n \in \mathbb{N}$, $\lim(x_n) = 3$.

Exercise 5

a) Show directly from the definition that a bounded, monotone decreasing sequence is a Cauchy sequence.

b) If $x_n := \sqrt{2n}$, show that (x_n) satisfies $\lim|x_{n+1} - x_n| = 0$, but that is not a Cauchy sequence.

Proof.

a) Let (x_n) be a bounded and monotone decreasing sequence. $\exists\, M \in \mathbb{R}$ such that $|x_n| \leq M \ \forall n \in \mathbb{N}$.
$x_n \geq -M \quad \forall n \in \mathbb{N}$. The set $\{x_n : n \in \mathbb{N}\}$ has an infimum. Let $x = \inf\{x_n : n \in \mathbb{N}\}$.
If $\varepsilon > 0$, let $H \in \mathbb{N}$ be such that

$$x \leq x_H < x + \varepsilon.$$

If $m \geq n \geq H$, then $x \leq x_m \leq x_n \leq x_H < x + \varepsilon$.
Hence, $|x_n - x_m| < \varepsilon$, $\forall\, m, n \geq H$,
that is, (x_n) is a Cauchy sequence.

b) Note that

$$|x_{n+1} - x_n| = \sqrt{2(n+1)} - \sqrt{2n}$$
$$= \frac{2(n+1) - 2n}{\sqrt{2(n+1)} + \sqrt{2n}}$$
$$= \frac{2}{\sqrt{2(n+1)} + \sqrt{2n}} < \frac{2}{\sqrt{2n}} < \frac{2}{\sqrt{n}}$$

Let $\varepsilon > 0$. Note that $\frac{2}{\sqrt{n}} < \varepsilon$ iff $n > \frac{4}{\varepsilon^2}$.

Choose $K > \frac{4}{\varepsilon^2}$, we have $|x_{n+1} - x_n| < \varepsilon, \forall n \geq K$.

So $\lim|x_{n+1} - x_n| = 0$.

To show that (x_n) is Not cauchy,

take $m = 4n$. We have

$$|x_n - x_m| = |x_{4n} - x_n|$$

$$= |\sqrt{8n} - \sqrt{2n}|$$

$$= 2\sqrt{2n} - \sqrt{2n}$$

$$= \sqrt{2n} > \sqrt{n}, \quad \forall n \in \mathbb{N}$$

If we take $\varepsilon = 1$, $\forall k \in \mathbb{N}$, $\exists n, m$ such that $|x_n - x_m| > \sqrt{n} > 1 = \varepsilon$.

Hence, (x_n) is not a Cauch sequence.