King Fahd University of Petroleum and Mineral
College of Computing and Mathematics
Department of Mathematics

MATH341 – Advanced Calculus I

Academic Year 2022-23

Term 221

Major Exam 2 Nov. 20, 2022

Name:	Solution
ID#:	

- ***The answers must be fully supported by logical arguments to get full credit***

Question	Score	Max Score
1		15
2		15
3		20
4		20
5		15
6		15
Total		**100**

Time allowed: **120** Minutes

Exercise 1

State

(a) The Sequential Criterion for Continuity
(b) The Discontinuity Criterion
(c) The Continuous Extension Theorem

Answer

(a) A function $f: A \to \mathbb{R}$ is continuous at $c \in A$ if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.

(b) Let $f: A \to \mathbb{R}$, and let $c \in A$. Then f is discontinuous at c if and only if there exists a sequence (x_n) in A that converges to c, but the sequence $(f(x_n))$ does not converge to $f(c)$

(c) A function f is uniformly continuous on the interval (a, b) if and only if it can be defined at the end points a and b such that the extended function is continuous on $[a, b]$

Exercise 2

(a) Show that the limit does not exist: $\lim_{x\to 0}(x + \mathrm{sgn}(x))$

(b) Let $f: \mathbb{R} \to \mathbb{R}$ be such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Assume that $\lim_{x\to 0} f(x) = L$ exists. Prove that $L = 0$, and show that f has a limit at every point $c \in \mathbb{R}$. Give an example of such functions.

(a) Proof. Let $f(x) = x + \mathrm{sgn}(x)$.
Consider the sequences $x_n = \frac{1}{n}$ and $y_n = -\frac{1}{n}$.

$\lim (x_n) = 0$ and $\lim(f(x_n)) = \lim\left(\frac{1}{n}+1\right)$
$= 1$

$\lim(y_n) = 0$ and $\lim(f(y_n)) = \lim\left(-\frac{1}{n}-1\right) = -1.$

We conclude that $\lim_{x\to 0}(x + \mathrm{sgn}(x))$ does not exist.

(b) Proof. Since $f(2x) = f(x) + f(x) = 2f(x)$, we have

$L = \lim_{x\to 0} f(x) = \lim_{x\to 0} f(2x) = \lim_{x\to 0} 2f(x) = 2L.$

We obtain $L = 0$.

Since $f(x) - f(c) = f(x-c)$, we have

$\lim_{x\to c}(f(x) - f(c)) = \lim_{x\to c} f(x-c) = \lim_{y\to 0} f(y) = 0.$

Hence $\lim_{x\to c} f(x) = f(c)$ $\forall c \in \mathbb{R}$.

f has a limit at every point $c \in \mathbb{R}$.

Example: $f(x) = x$, $\lim_{x\to 0} f(x) = 0$.
$\Rightarrow f(x+y) = x + y = f(x) + f(y).$

f is continuous at every point $c \in \mathbb{R}$

Exercise 3

(a) Let $K > 0$ and let $f: \mathbb{R} \to \mathbb{R}$ satisfy the condition $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}$. Show that f is continuous at every point $c \in \mathbb{R}$.

(b) Give an example of a function $f: [0,1] \to \mathbb{R}$ that is discontinuous at every point of $[0,1]$ but such that $|f|$ is continuous on $[0,1]$.

(a) Let $c \in \mathbb{R}$ be given and let $\varepsilon > 0$. choose $\delta = \varepsilon/K$.

If $0 < |x-c| < \delta$, then $|f(x)-f(c)| \leq K|x-c| < K(\varepsilon/K) = \varepsilon$.

Hence, f is continuous every point $c \in \mathbb{R}$.

(b) Let $f(x) = 1$ if x is rational, and $f(x) = -1$ if x is irrational.

If $c \in [0,1]$ is a rational number, let (x_n) be a sequence of irrational numbers that converges to c.
Since $f(x_n) = -1$ $\forall n \in \mathbb{N}$, we have $\lim(f(x_n)) = -1$, while $f(c) = 1$. Hence, f is discontinuous at c.
If $b \in [0,1]$ is an irrational number, let (y_n) be a sequence of rational numbers that converges to b.
Since $f(y_n) = 1$ $\forall n \in \mathbb{N}$, we have $\lim f(y_n) = 1$, while $f(b) = -1$. Thus, f is discontinuous at b.

Therefore, f is not continuous on $[0,1]$.
$|f(x)| = 1$ $\forall x \in [0,1]$, $|f|$ is continuous on $[0,1]$.

Exercise 4

(a) Show that the function $f(x) = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

(b) Show that the function $f(x) = 1/x^2$ is not uniformly continuous on $A = (0, \infty)$.

(a) Let $\varepsilon > 0$. If $x, y \in \mathbb{R}$, then

$$|f(x) - f(y)| = \left| \frac{1}{1+x^2} - \frac{1}{1+y^2} \right|$$

$$= \left| \frac{1+y^2 - (1+x^2)}{(1+x^2)(1+y^2)} \right| = \left| \frac{y^2 - x^2}{(1+x^2)(1+y^2)} \right|$$

$$= \left| \frac{(y-x)(y+x)}{(1+x^2)(1+y^2)} \right| \leq \frac{|x| + |y|}{(1+x^2)(1+y^2)} |x-y|$$

$$\leq \left(\frac{1}{2} + \frac{1}{2}\right) |x-y| = |x-y|$$

Take $\delta = \varepsilon$. If $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Hence, f is uniformly continuous on $\mathbb{R}$.

(b) Let $x_n = 1/n$ and $y_n = \frac{1}{n+1}$. Take $\varepsilon_0 = \frac{1}{2}$.

$$\lim |x_n - y_n| = \lim \left| \frac{1}{n} - \frac{1}{n+1} \right| = \lim \left| \frac{n+1-n}{n(n+1)} \right|$$

$$= \lim \left| \frac{1}{n(n+1)} \right| = 0.$$

We have

$$|f(x_n) - f(y_n)| = |n^2 - (n+1)^2| = |n^2 - n^2 - 2n - 1|$$

$$= 2n+1 \geq 1 > \varepsilon_0 \quad \forall n \in \mathbb{N}.$$

Hence, f is not uniformly continuous on $A = (0, \infty)$.

Exercise 5 Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at c and that $f(c) = 0$. Show that $g(x) = |f(x)|$ is differentiable at c if and only if $f'(c) = 0$.

Proof. By definition

$$g'(c) = \lim_{h\to 0} \frac{|f(c+h)| - |f(c)|}{h} = \lim_{h\to 0} \frac{|f(c+h)|}{h},$$

if the limit exists.

($\Rightarrow$) Suppose, on the contrary, that $f'(c) = L \neq 0$.

We have

$$\lim \frac{f(c \pm \frac{1}{n})}{\pm\frac{1}{n}} = L,$$

while

$$\lim \frac{|f(c \pm \frac{1}{n})|}{\pm\frac{1}{n}} = \pm L,$$

So that $|f|'(c)$ does not exist.

Hence, $f'(c) = 0$.

($\Leftarrow$) $$0 = |f'(c)| = \lim_{h\to 0} \frac{|f(c+h)|}{h}.$$

It follows that $g'(c) = 0$.

g is differentiable at c.

Exercise 6

(a) Prove that $|\sin x - \sin y| \leq |x - y|$ for all x, y in $\mathbb{R}$.

(b) Let f, g be differentiable on $\mathbb{R}$ and suppose that $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x \geq 0$. Show that $f(x) \leq g(x)$ for all $x \geq 0$.

(a) Let $f(x) = \sin x$. Since f is differentiable everywhere,

If $x < y$, then $\exists\, c \in (x, y)$ such that

$$|f(x) - f(y)| = |f'(c)|\,|x-y|.$$

Therefore, $|\sin x - \sin y| \leq |\cos c|\,|x-y| \leq |x-y|$

(b) Let $h = g - f$. We have $h(0) = g(0) - f(0) = 0$ and $h'(x) \geq 0, \forall x \in \mathbb{R}$.

Since f and g are both differentiable on $\mathbb{R}$,

$\exists\, c \in (0, x)$ such that

$$h(x) - h(0) = h'(c)\,(x - 0)$$

We have $h(x) = h'(c)\,x \geq 0$,

that is $g(x) - f(x) \geq 0$.

Thus, $f(x) \leq g(x) \quad \forall x \in \mathbb{R}$.