King Fahd University of Petroleum and Mineral
College of Computing and Mathematics
Department of Mathematics

MATH341 – Advanced Calculus I

Academic Year 2022-23 Term 221

Final Exam

Time allowed: **150** Minutes

Name:	
ID#:	

- ***The answers must be fully supported by logical arguments to get full credit***

Question	Score	Max Score
1		12
2		18
3		17
4		12
5		12
6		14
7		21
8		14
Total		**120**

Question 1

(a) Show directly from the definition that the sequence $\left(\frac{n+1}{n}\right)$ is a Cauchy sequence.

(b) Let (x_n) be a Cauchy sequence such that x_n is an integer for every $n \in \mathbb{N}$. Show that (x_n) is ultimately constant.

Question 2

(a) Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$ be continuous at a point $c \in A$. Show that for any $\varepsilon > 0$, there exists a neighborhood $V_\delta(c)$ of c such that if $x, y \in A \cap V_\delta(c)$, then $|f(x) - f(y)| < \varepsilon$.

(b) Show that $f(x) = 1/x^2$ is uniformly continuous on $A = [1, \infty)$, but that it is not uniformly continuous on $B = (0, \infty)$.

Question 3

a) Show that $f(x) = x^{1/3}$, $x \in \mathbb{R}$, is not differentiable at $x = 0$.

b) Show that if $x \geq 0$, then

$$1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$$

Question 4

(a) Suppose that f and g are continuous on $[a, b]$, differentiable on (a, b), that $c \in [a, b]$ and that $g(x) \neq 0$ for $x \in [a, b],\ x \neq c$. Let $\alpha = \lim_{x \to c} f$ and $\beta = \lim_{x \to c} g$. If $\beta = 0$, and if $\lim_{x \to c} \frac{f(x)}{g(x)}$ exists in $\mathbb{R}$, show that $\alpha = 0$.

(b) Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$, let $f(0) = 0$, and let $g(x) = \sin x$ for $x \in \mathbb{R}$. Show that $\lim_{x \to 0} \frac{f(x)}{g(x)} = 0$ but that $\lim_{x \to 0} \frac{f'(x)}{g'(x)}$ does not exist.

Question 5

(a) If $f \in \mathcal{R}[a,b]$ and $|f(x)| \leq M$ for all $x \in [a,b]$, show that $\left|\int_a^b f\right| \leq M(b-a)$.

(b) If f and g are continuous on $[a,b]$ and if $\int_a^b f = \int_a^b g$, prove that there exists $c \in [a,b]$ such that $f(c) = g(c)$.

Question 6

(a) Show there does not exist a continuously differentiable function f on $[0,2]$ such that $f(0) = -1, f(2) = 4$, and $f'(x) \leq 2$ for $0 \leq x \leq 2$. [Apply the Fundamental Theorem]

(b) If $g(x) = x$ for $|x| \geq 1$ and $g(x) = -x$ for $|x| < 1$ and if $G(x) = \frac{1}{2}|x^2 - 1|$, show that

$$\int_{-2}^{3} g(x)dx = G(3) - G(-2) = 5/2.$$

Question 7

(a) Show that the series $\frac{1}{1^2}+\frac{1}{2^3}+\frac{1}{3^2}+\frac{1}{4^3}+\cdots$ is convergent, but that the Root Test fails to apply.

(b) If (a_n) is a bounded decreasing sequence and (b_n) is a bounded increasing sequence and if $x_n = a_n + b_n$ for $n \in \mathbb{N}$, show that $\sum_{n=1}^{\infty} |x_n - x_{n+1}|$ is convergent.

Question 8

(a) Find a series expansion for $\int_0^x e^{-t^2} dt$ for $x \in \mathbb{R}$.

(b) Show that if $\sum a_n$ is an absolutely convergent series, then the series $\sum a_n \sin nx$ is absolutely and uniformly convergent.