King Fahd University of Petroleum and Minerals
Department of Mathematics
MATH341 - Advanced Calculus I
Major I Exam – Semester 241

Exercise 1

Show that there exists a positive real number u such that $u^3 = 2$.

Exercise 2

(i) Prove that if $\lim(x_n) = x$ and if $x > 0$, then there exists a natural number M such that $x_n > 0$ for all $n \geq M$.

(ii) Use the definition of the limit of a sequence to establish $\lim\limits_{n\to\infty} \dfrac{2n}{n+1} = 2$.

(iii) Let (x_n) be a sequence of positive real numbers such that $\lim\limits_{n\to\infty} (x_n^{1/n}) < 1$. Show that $\lim(x_n) = 0$.

Exercise 3

Let $X = (x_n)$ be a sequence of positive real numbers such that $L := \lim_{n \to \infty} \frac{x_{n+1}}{x_n}$ exists.

(i) Show that , if $L < 1$, then (x_n) converges and $\lim x_n = 0$.

(ii) What about the case $L = 1$?

(iii) Show that if $L > 1$, then X is not a bounded sequence and hence is not convergent.

Exercise 4

(i) Let $a > 0$ and let $z_1 > 0$. Define $z_{n+1} := \sqrt{a + z_n}$ for $n \in \mathbb{N}$. Show that (z_n) converges and find its limit.

(ii) Let $x_n := \frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2}$ for each $n \in \mathbb{N}$. Prove that (x_n) converges.

Exercise 5

(i) Show the divergence of the sequence $\sin(\frac{n\pi}{4})$.

(ii) Suppose that $x_n > 0$ for all $n \in \mathbb{N}$ and that $\lim((-1)^n x_n)$ exists. Show that (x_n) converges

Exercise 6

(i) If $x_1 = 2$ and $x_{n+1} := 2 + \dfrac{1}{x_n}$ for $n \geq 1$, show that (x_n) is a contractive sequence and find its limit.

(ii) Let (x_n) be a Cauchy sequence such that x_n is an integer for every $n \in \mathbb{N}$. Show that (x_n) is ultimately constant.

Exercise 7

Let (x_n) and (y_n) be sequences of positive numbers such that $\lim(x_n/y_n) = 0$.

(a) Show that if $\lim(x_n) = \infty$, then $\lim(y_n) = \infty$.

(b) Show that if (y_n) is bounded, then $\lim(x_n) = 0$.