King Fahd University of Petroleum and Minerals
Department of Mathematics
MATH 341 - Advanced Calculus I
Major Exam II – Semester 241

Exercise 1

1. Use the ϵ-δ definition of limit to establish the following $\lim_{x \to 1} \frac{x}{1+x^2} = \frac{1}{2}$.
2. Let I be an interval in $\mathbb{R}$, let $f : I \to \mathbb{R}$, and let $c \in I$. Suppose there exist constants K and L such that

$$|f(x) - L| < K|x - c| \quad \text{for } x \in I.$$

   Show that $\lim_{x \to c} f(x) = L$.

Exercise 2

1. Find $\lim_{x \to 0} \dfrac{\sqrt{1+2x} - \sqrt{1+3x}}{x+2x^2}$ where $x > 0$.
2. Prove that $\lim_{x \to 0} \cos\left(\dfrac{1}{x}\right)$ does not exist, but that $\lim_{x \to 0} x \cos\left(\dfrac{1}{x}\right) = 0$.

Exercise 3

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at c and let $f(c) > 0$. Show that there exists a neighborhood $V(c)$ of c such that if $x \in V(c)$, then $f(x) > 0$.
2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $f(r) = 0$ for every rational number r. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.
3. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 2x & \text{if } x \text{ is rational} \\ x + 3 & \text{if } x \text{ is irrational} \end{cases}$$

   Find all points at which g is continuous.

Exercise 4

1. If f and g are continuous on $\mathbb{R}$, let $S = \{x \in \mathbb{R} : f(x) \geq g(x)\}$. If $s_n \in S$ and $\lim_{n\to\infty} s_n = s$, show that $s \in S$.
2. Let $h : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ satisfying $h(m/3^n) = 0$ for all $m \in \mathbb{Z}, n \in \mathbb{N}$. Show that $h(x) = 0$ for all $x \in \mathbb{R}$.

Exercise 5

1. Let $I = [a, b]$ be an interval, and let $f : I \to \mathbb{R}$ be continuous on I, and assume that $f(a) < 0$ and $f(b) > 0$. Let

   $$W = \{x \in I : f(x) < 0\},$$

   and let $w = \sup W$. Prove that $f(w) = 0$.

2. Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for each $x \in I$. Prove that there exists a number $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in I$.

Exercise 6

1. Show that the function $f(x) = \frac{1}{x^2}$ is uniformly continuous on $A = [1, \infty)$, but that it is not uniformly continuous on $B = (0, \infty)$.
2. If f is uniformly continuous on $A \subset \mathbb{R}$, and $|f(x)| \geq k > 0$ for all $x \in A$, show that $\dfrac{1}{f}$ is uniformly continuous on A.
3. Prove that if f is uniformly continuous on a bounded subset A of $\mathbb{R}$, then f is bounded on A.

Exercise 7

1. Show that if f and g are positive increasing functions on an interval I, then their product fg is increasing on I.
2. Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be increasing on I. Suppose that $c \in I$ is not an endpoint of I. Then the following statements are equivalent:
   (a) f is continuous at c.
   (b) $\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f$.
   (c) $\sup\{f(x) : x \in I, x < c\} = f(c) = \inf\{f(x) : x \in I, x > c\}$.