King Fahd University of Petroleum and Minerals
Department of Mathematics
MATH 341 - Advanced Calculus I
Final Exam – Semester 241

Exercise 1

1. If $r > 0$ is a rational number, let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^r \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

   Determine those values of f for which $f'(0)$ exists.

2. Use the Mean Value Theorem to prove that

$$\frac{x-1}{x} < \ln x < x - 1 \quad \text{for } x > 1.$$

3. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b). Show that if $\lim_{x \to a} f'(x) = A$, then $f'(a)$ exists and equals A. **Hint:** Use the definition of $f'(a)$ and the Mean Value Theorem.

Exercise 2

1. Let I be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. Show that if f' is positive on I, then f is strictly increasing on I.
2. Let I be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. Show that if the derivative f' is never 0 on I, then either $f'(x) > 0$ for all $x \in I$ or $f'(x) < 0$ for all $x \in I$.
3. Let I be an interval. Prove that if f is differentiable on I and if the derivative f' is bounded on I, then f satisfies a Lipschitz condition on I.

Exercise 3

Evaluate the following limits:

(a) $\lim_{x \to 0^+} x^{2x}$

(b) $\lim_{x \to \infty} (1 + 3/x)^x$

(c) $\lim_{x \to 0} (\frac{1}{x} - \frac{1}{\arctan x})$

Exercise 4

1. Show that if $x > 0$, then
$$1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x.$$
2. If $g(x) := \sin x$, show that the remainder term in Taylor's Theorem converges to 0 as $n \to \infty$ for each fixed x and x_0
3. Suppose that $I \subseteq \mathbb{R}$ is an open interval and that $f''(x) > 0$ for all $x \in I$. If $c \in I$, show that the graph of f on I is above the tangent line to the graph at $(c, f(c))$.

Exercise 5

Let $0 < a < b$, let $f(x) := x^2$ for $x \in [a, b]$ and let $\mathcal{P} := \{x_i\}_{i=0}^n$ be a partition of $[a, b]$. For each i, let q_i be the square root of

$$\frac{1}{3}\left(x_i^2 + x_i x_{i-1} + x_{i-1}^2\right).$$

(a) Show that q_i satisfies $0 \le x_{i-1} \le q_i \le x_i$.

(b) Show that $f(q_i)(x_i - x_{i-1}) = \frac{1}{3}(x_i^3 - x_{i-1}^3)$.

(c) If $\dot{\mathcal{Q}}$ is the tagged partition with the same subintervals as $\mathcal{P}$ and the tags q_i, show that

$$S(f; \dot{\mathcal{Q}}) = \frac{1}{3}(b^3 - a^3).$$

(d) Show that $f \in \mathcal{R}[a, b]$ and

$$\int_a^b x^2\, dx = \frac{1}{3}(b^3 - a^3).$$

Exercise 6

Let $f, g \in \mathcal{R}[a,b]$.

(a) If $t \in \mathbb{R}$, show that $\int_a^b (tf \pm g)^2 \geq 0$.

(b) Use (a) to show that $2|\int_a^b fg| \leq t\int_a^b f^2 + \left(\frac{1}{t}\right)\int_a^b g^2 \quad$ for $t > 0$.

(c) If $\int_a^b f^2 = 0$, show that $\int_a^b fg = 0$.

(d) Prove that

$$\left|\int_a^b fg\right|^2 \leq \left(\int_a^b f^2\right)\left(\int_a^b g^2\right).$$

This inequality is called the Cauchy-Schwarz Inequality.

Exercise 7

1. Show that $\lim_{n\to\infty} \dfrac{x}{x+n} = 0$ for all $x \geq 0$.
2. Show that if $a > 0$, then the convergence of the sequence $f_n(x) = \dfrac{x}{x+n}$ is uniform on the interval $[0, a]$, but is not uniform on the interval $[0, \infty)$.