King Fahd University of Petroleum & Minerals
Department of Mathematics
Math 371
Major Exam 1 (Term 211)

Time: 90 Minutes **Marks: 60**

Name: ______________________________ ID#: _______________

Instructor: ______________________ Sec #: _______ Serial #: _______

- Answers should be neat, clear, and legible.
- Show all steps.
- Use of Calculator is allowed.
- Mobiles are not allowed.

Question #	Marks	Maximum Marks
1		10
2		10
3		10
4		10
5		10
6		10
Total		60

Q#1 (8+2 points) Given that $f(x) = e^x \cos x$.

(a) Find the second Taylor polynomial $P_2(x)$ about $x_0 = 0$.

(b) Find the error $|f(0.1) - P_2(0.1)|$.

Given:

Solution: (a) $f(x) = e^x \cos x, \quad x_0 = 0$

$P_2(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)(x - x_0)^2$

$= f(0) + x f'(0) + \frac{x^2}{2!} f''(0)$ 2

Now, $f'(x) = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x)$ 1

$f''(x) = e^x(\cos x - \sin x) + e^x(-\sin x - \cos x)$

$= e^x(\cancel{\cos x} - \sin x - \sin x - \cancel{\cos x})$ 1

$= -2e^x \sin x$

$\therefore f(0) = e^0 \cos(0) = 1$ 1

$f'(0) = e^0(\cos(0) - \sin(0)) = 1(1-0) = 1$ 1

$f''(0) = -2e^0 \sin(0) = -2(1)(0) = 0$ 1

Thus, $P_2(x) = 1 + x$ 1

(b) $f(0.1) = e^{0.1}\cos(0.1) = 1.0996$

$P_2(0.1) = 1 + 0.1 = 1.1$ 1

Thus, $|f(0.1) - P_2(0.1)| = |1.0996 - 1.1| = 0.0004$ 1

Q#2 (5+5 points) **(a)** Define a function having $\sqrt{3}$ as a root and use it with the Bisection Method (take smallest interval having integer end-points) to find p_1, p_2, p_3 (use 4 decimal places in calculations).

(b) Estimate the number of iterations required to achieve 10^{-5} accuracy with the interval used in part (a).

Solution: (a) $f(x) = x^2 - 3$ (1)

$f(1) = (1)^2 - 3 = -2 < 0$

$f(2) = (2)^2 - 3 = 4 - 3 = 1 > 0$ (1)

Thus, a root $p \in [1, 2]$. Now, $a = 1, b = 2$,
Using Bisection Method:

$$p_n = \frac{a_n + b_n}{2}$$ (1)

$$\Rightarrow p_1 = \frac{1+2}{2} = \frac{3}{2} = 1.5, \quad f(p_1) = f(1.5) = -0.75 < 0$$

$$\Rightarrow p_2 = \frac{1.5+2}{2} = 1.75, \quad f(1.75) = 0.0625 > 0$$ (1)

$$\Rightarrow p_3 = \frac{1.5+1.75}{2} = 1.625$$ (1)

(b)

$$|p_n - p| \le \frac{b-a}{2^n} < 10^{-5}$$ (1)

$$\Rightarrow \frac{2-1}{2^n} < 10^{-5}$$ (1)

or $$2^{-n} < 10^{-5}$$ (1)

$$-n \log_{10} 2 < -5 \log_{10} 10$$

$$n > \frac{5}{\log_{10} 2} \approx 16.61$$ (1)

Thus, $n = 17$. (i.e, at least 17 iterations required to achieve 10^{-5} accuracy). (1)

Q#3 (10 points) Given that $g(x) = \pi + \frac{1}{2}sin(\frac{x}{2})$ has a unique fixed point in the interval $[0, 2\pi]$.

Estimate the number of iterations required to achieve 10^{-2} accuracy with $p_0 = 0$.

Solution: $g(x) = \pi + \frac{1}{2}\sin(\frac{x}{2})$, $a = 0$, $b = 2\pi$.

$g'(x) = \frac{1}{4}\cos(\frac{x}{2})$ (1)

$|g'(x)| = |\frac{1}{4}\cos(\frac{x}{2})| = \frac{1}{4}|\cos\frac{x}{2}| \le \frac{1}{4} = K < 1$ (2)

$(\because |\cos\theta| \le 1)$

$|P_n - P| \le K^n \max\{P_0 - a, b - P_0\} < 10^{-2}$ (1)

$\Rightarrow (\frac{1}{4})^n \max\{0 - 0, 2\pi - 0\} < 10^{-2}$

$\Rightarrow (\frac{1}{4})^n \cdot 2\pi < 10^{-2}$ (1)

or $(\frac{1}{4})^n < \frac{10^{-2}}{2\pi}$ (1)

or $n(\ln 1 - \ln 4) < -\ln(2\pi) - 2\ln 10$

or $-n\ln 4 < -\ln(2\pi) - 2\ln 10$ (1)

or $n\ln 4 > \ln(2\pi) + 2\ln 10$ (1)

or $n > \frac{\ln(2\pi) + 2\ln 10}{\ln 4} = 4.64$ (1)

Thus, $n = 5$, i.e, 5 number of iterations required to achieve 10^{-2} accuracy with $P_0 = 0$. (1)

Q#4 (10 points) For $sin\, x = e^{-x}$,

use Newton's method to find a solution (root) with in 10^{-2} with $p_0 = 0.5$ (use 4 decimal places in calculations).

Given:

Solution: $\sin x = e^{-x}$, $P_0 = 0.5$

$\Rightarrow f(x) = \sin x - e^{-x}$ 2 (or $f(x) = e^{-x} - \sin x$)

$\Rightarrow f'(x) = \cos x + e^{-x}$ 1

Newton's iterative formula:

$$P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})}$$

$n=1$: $P_1 = P_0 - \frac{f(P_0)}{f'(P_0)} = 0.5 - \frac{\sin(0.5) - e^{-0.5}}{\cos(0.5) + e^{-0.5}} = 0.5856$ 2

$n=2$: $P_2 = P_1 - \frac{f(P_1)}{f'(P_1)} = 0.5856 - \frac{\sin(0.5856) - e^{-0.5856}}{\cos(0.5856) + e^{-0.5856}}$ 2

$\Rightarrow P_2 = 0.5885$

Now,

$|P_1 - P_0| = |0.5856 - 0.5| = 0.0856 > 10^{-2}$ 1

$|P_2 - P_1| = |0.5885 - 0.5856| = 0.0029 < 10^{-2}$ 1

Thus, $P_2 = 0.5885$ is the required root with in 10^{-2} accuracy. 1

Q#5 (7+3 points) Let $f(0.25) = 1.65,\ f(0.5) = 2.72,\ f(0.75) = 4.49$ (use 4 decimal places in calculations).

(a) Construct a Second degree Lagrange interpolating polynomial.

(b) Use this Second degree Lagrange interpolating polynomial to approximate $f(0.43)$.

Solution: Given:

$x_0 = 0.25,\quad x_1 = 0.5,\quad x_2 = 0.75$

$f(x_0) = 1.65,\quad f(x_1) = 2.72,\quad f(x_2) = 4.49$

(a) $P_2(x) = L_0(x)\,f(x_0) + L_1(x)\,f(x_1) + L_2(x)\,f(x_2)$

Now, 2

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0.5)(x-0.75)}{(0.25-0.5)(0.25-0.75)} = \frac{}{0.125}$$

2

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0.25)(x-0.75)}{(0.5-0.25)(0.5-0.75)} = \frac{}{-0.0625}$$

2

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0.25)(x-0.5)}{(0.75-0.25)(0.75-0.5)} = \frac{}{0.125}$$

1

$$\therefore P_2(x) = (1.65)\frac{(x-0.5)(x-0.75)}{0.125} - (2.72)\frac{(x-0.25)(x-0.75)}{0.0625} + (4.49)\frac{(x-0.25)(x-0.5)}{0.125}$$

(b) 1

$$\boxed{f(0.43) \approx P_2(0.43)}$$

$$\therefore P_2(0.43) = (1.65)\frac{(0.43-0.5)(0.43-0.75)}{0.125} - (2.72)\frac{(0.43-0.25)(0.43-0.75)}{0.0625}$$

$$+ (4.49)\frac{(0.43-0.25)(0.43-0.5)}{0.125} \quad 2$$

$\Rightarrow P_2(x) = 2.3498$

Thus, $f(0.43) = 2.3498$.

Q#6 (7+3 points) Consider $f(x) = e^x$ and the nodes $x_0 = 1$, $x_1 = 2$ and $x_2 = 3$. (use 4 decimal places in calculations)

(a) Use Netown's divided differences to construct the interpolating polynomial for $f(x)$.

(b) Use this interpolating polynomial to approximate $e^{2.10}$.

Solution: $f(x) = e^x$, $x_0 = 1$, $x_1 = 2$, $x_2 = 3$

(a) $\boxed{\text{Newton's divided differences}}$ Interpolating polynomial with

$$P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \quad 1$$

where

$$a_0 = f[x_0] = f(x_0) \quad 1$$

$$a_1 = f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad 1$$

$$a_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \quad 1$$

Now

$$\left.\begin{aligned} a_0 &= f(x_0) = e^1 = 2.7183 \\ a_1 &= \frac{e^2 - e^1}{2 - 1} = 4.6708 \end{aligned}\right\}$$

$$\left.\begin{aligned} f[x_1, x_2] &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{e^3 - e^2}{3 - 2} = 12.6965 \\ \therefore a_2 &= \frac{12.6965 - 4.6708}{3 - 1} = 4.0129 \end{aligned}\right\} 1$$

Thus, $P_2(x) = 2.7183 + 4.6708(x - 1) + 4.0129(x - 1)(x - 2)$ $= 2$

(b) $\boxed{f(2.10) \approx P_2(2.10)}$ $1 + 2$ $= 8.2976 = \big(2.7183 + 4.6708(2.1 - 1) + 4.0129(2.1 - 1)(2.1 - 2)\big)$

(END)