King Fahd University of Petroleum & Minerals
Department of Mathematics
Math 371
Major Exam 2 (Term 211)

Time: 90 Minutes **Marks: 60**

KEY

Name: ______________________ ID#: ______________

Instructor: ______________________ Sec #: ________ Serial #: ________

- Answers should be neat, clear, and legible.
- Show all steps.
- Use of Calculator is allowed.
- Mobiles are not allowed.

Question #	Marks	Maximum Marks
1		10
2		10
3		8
4		8
5		8
6		8
7		8
Total		60

Q#1 [2+8 points] **(a)** Given some experimental data points $(x_i, y_i), i = 1, \cdots, n$. What is the **quantity** that the linear least squares method is minimizing?

(b) Construct the least squares polynomial of degree one and compute the error for the following data.

x_i:	4.0	4.2	4.5	4.7	5.1	5.5
y_i:	102.56	113.18	130.11	142.05	167.53	195.14

Solution:

(a) $$\sum_{i=1}^{n} [y_i - (a_1 x_i + a_0)]^2 \quad \text{or} \quad \sum_{i=1}^{n} [y_i - P_1(x_i)]^2$$ 2

(b) 3

x_i	y_i	x_i^2	$x_i y_i$	$P(x_i) = 61.6992x - 146.1677$
4.0	102.56			100.6291
4.2	113.18			112.9689
4.5	130.11			131.4787
4.7	142.05			143.8185
5.1	167.53			168.4982
5.5	195.14			193.1779
Σ: 28	850.57	132.24	4066.4	$E = \sum_{i=1}^{6} [y_i - P(x_i)]^2 = 13.5611$

$$a_0 = \frac{(132.24)(850.57) - (4066.4)(28)}{6(132.24) - (28)^2} = -146.1677$$ 1

$$a_1 = \frac{6(4066.4) - (28)(850.57)}{6(132.24) - (28)^2} = 61.6992$$ 1

Linear least squares polynomial is

$$P(x) = a_1 x + a_0 = 61.6992x - 146.1677.$$ 1

$$\text{Error} = \sum_{i=1}^{6} [y_i - P(x_i)]^2 = (102.56 - 100.6291)^2 + \cdots + (195.14 - 193.1779)^2$$
$$= 13.5611$$ 2

Q#2 [10 points] A clamped cubic spline $S(x)$ for a function $f(x)$ is defined on $[1,3]$ by

$$S(x) = \begin{cases} S_0(x) = 3(x-1) + 2(x-1)^2 - (x-1)^3, & \text{if } 1 \le x < 2 \\ S_1(x) = A + B(x-2) + C(x-2)^2 + D(x-2)^3, & \text{if } 2 \le x \le 3 \end{cases}$$

Given that $f'(1) = f'(3)$; Find A, B, C, and D.

Solution: Here $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $f'(1) = f'(3)$. 1

(b) $S_0(x_0) = f(x_0) \Rightarrow S_0(1) = f(1) \Rightarrow \boxed{0 = f(1)}$

$S_0(x_1) = f(x_1) \Rightarrow 3 + 2 - 1 = f(2) \Rightarrow \boxed{f(2) = 4}$ 1

$S_1(x_1) = f(x_1) \Rightarrow S_1(2) = f(2) \Rightarrow \boxed{A = 4}$ 1

$S_1(x_2) = f(x_2) \Rightarrow S_1(3) = f(3) \Rightarrow A + B + C + D = f(3)$

(d) $S_0'(x) = 3 + 4(x-1) - 3(x-1)^2$

$S_1'(x) = B + 2C(x-2) + 3D(x-2)^2$ 2

$S_1'(2) = S_0'(2) \Rightarrow B = 3 + 4 - 3 \Rightarrow \boxed{B = 4}$

(e) $S_0''(x) = 4 - 6(x-1)$

$S_1''(x) = 2C + 6D(x-2)$

$S_1''(2) = S_0''(2) \Rightarrow 4 - 6 = 2C$

$\Rightarrow \boxed{C = -1}$ 2

(f) Clamped BC:

$S'(1) = f'(1)$, $S'(3) = f'(3)$

$\Rightarrow 3 = f'(1)$, $B + 2C + 3D = f'(3)$

Given $f'(1) = f'(3)$ 3

$\Rightarrow B + 2C + 3D = 3$

$\boxed{D = \frac{1}{3}}$ $(\because C = -1, B = 4)$

Thus, $A = 4$, $B = 4$, $C = -1$, $D = \frac{1}{3}$.

Q#3 [8 points] Determine the values of n and h required to approximate $\int_2^3 x\,lnx\,dx$ to within 10^{-6}, using the Composite Simpson's Rule.

Solution: We want n and h such that

$$\left|\frac{b-a}{180}h^4 f^{(4)}(\mu)\right| < 10^{-6} \quad \text{——— } (*) \qquad 1$$

Here $a=2$, $b=3$, $h=\frac{b-a}{n}=\frac{1}{n}$

$$f(x) = x\ln x$$

$$f'(x) = x\cdot\frac{1}{x}+\ln x = 1+\ln x$$

$$f''(x) = \frac{1}{x}$$

$$f'''(x) = -\frac{1}{x^2}$$

$$f^{(4)}(x) = \frac{2}{x^3}$$

1+1+1

$$\therefore \max_{2\le\mu\le3}\left|f^{(4)}(\mu)\right| = \left|\frac{2}{2^3}\right| = \left|\frac{2}{8}\right| = \frac{1}{4} \text{ or } \left|f^{(4)}(\mu)\right| \le \frac{1}{4}. \qquad 1$$

$$\therefore (*) \Rightarrow \left|\frac{3-2}{180}\left(\frac{1}{n}\right)^4 f^{(4)}(\mu)\right| < 10^{-6}$$

$$\Rightarrow \frac{1}{180}\cdot\frac{1}{n^4}\cdot\frac{1}{4} < 10^{-6}$$

$$\text{or} \quad n^4 > \left(\frac{1}{180\times4}\right)\times10^6 = 1388.9$$

$$n > (1388.9)^{1/4} \approx 6.10$$

1 + 1

Thus, $\boxed{n=8}$, $\boxed{h=\frac{1}{8}}$ for Composite Simpson's rule 1

Q#4 [4+4 points] **(a)** Use the most accurate three-point formula to determine the value $y'(8.3)$ by using the following data:

$$(8.1, 16.9441), (8.3, 17.56492), (8.5, 18.19056), \text{ and } (8.7, 18.82091).$$

(b) Compute the actual error and error bound (in Part (a)) if $f(x) = x\ln x$.

Solution: (a) $f'(x_0) \approx \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right]$, $h = 0.2$ → 1

$\therefore\ y'(8.3) \approx \frac{1}{2(0.2)}\left[f(8.3+0.2) - f(8.3-0.2)\right]$ → 1

$= \frac{1}{0.4}\left[f(8.5) - f(8.1)\right]$

$= \frac{1}{0.4}\left[18.19056 - 16.9441\right] = 3.1162.$ → 2

$\Rightarrow\ y'(8.3) \approx 3.1162.$

(b) Error Bound $= \frac{h^2}{6}|f'''(\xi)|$, ξ is in between 8.1 and 8.5 → 1

$f(x) = x\ln x$, $h = 0.2$

$f'(x) = x \cdot \frac{1}{x} + \ln x = 1 + \ln x$

$f''(x) = \frac{1}{x}$

$f'''(x) = -\frac{1}{x^2}$

$\therefore\ \max_{8.1 \le x \le 8.5} |f'''(x)| = \left|-\frac{1}{(8.1)^2}\right| = 0.0152$ → 1

$\therefore$ Error bound $= \frac{(0.2)^2}{6}(0.0152) = 0.00010133$ or 1.0133e-4

Exact: $f'(8.3)$ or $y'(8.3) = 1 + \ln(8.3) \approx 3.1163.$ → 1

Actual error $= |3.1163 - 3.1162| = 0.0001.$ → 1

$\Rightarrow$ Actual error < Error bound.

Q#5 [8 points] The Trapezoidal rule applied to $\int_0^4 f(x)\,dx$ gives the value 4 and Simpson's rule gives the value 2. What is $f(2)$?

Solution: Trapezoidal rule:

$$\frac{h}{2}[f(0)+f(4)] = 4 \longrightarrow 1$$

where $h = \frac{4-0}{1} = 4 \longrightarrow 1$

$$\therefore \quad f(0)+f(4) = 2 \quad \text{— (i)} \longrightarrow 1$$

Simpson's rule:

$$\frac{h}{3}[f(0)+4f(2)+f(4)] = 2 \longrightarrow 1$$

where $h = \frac{4-0}{2} = 2 \longrightarrow 1$

$$\therefore \quad \frac{\cancel{2}}{3}[2+4f(2)] = \cancel{2} \quad (\text{using (i)}) \quad 1$$

or $\quad 2+4f(2) = 3$

or $\quad f(2) = \frac{3-2}{4}$

$$\Rightarrow \boxed{f(2) = \frac{1}{4}}$$

2

Q#6 [8 points] Use Theorem 5.4 to show that the initial-value problem

$$\frac{dy}{dt} = \frac{4t^3}{1+t^4}\,y, \qquad 0 \le t \le 1, \qquad y(0) = 1,$$

has a unique solution.

Solution: $f(t,y) = \dfrac{4t^3}{1+t^4}\,y, \quad 0 \le t \le 1, \quad y(0) = 1$ 1

$$\left|\frac{\partial f}{\partial y}\right| = \left|\frac{4t^3}{1+t^4}\right| \le \frac{4(1)^3}{1+(1)^4} = \frac{4}{2} = 2 = L$$ 4

or let $g(t) = \dfrac{4t^3}{1+t^4}$

$$g'(t) = \frac{(1+t^4)12t^2 - 4t^3(0+4t^3)}{(1+t^4)^2} = \frac{12t^2+12t^6-16t^6}{(1+t^4)^2}$$

$$\Rightarrow g'(t) = \frac{12t^2-4t^6}{(1+t^4)^2} = \frac{4t^2(3-t^4)}{(1+t^4)^2}$$

$g'(t) = 0$

$\Rightarrow 4t^2(3-t^4) = 0$

$t = 0, \; t^4 = 3 \Rightarrow t = (3)^{1/4} \notin [0,1]$

Thus, $t=0, \; t=1 \Rightarrow \max|g(t)| = \dfrac{4(1)^3}{1+(1)^4} = 2$

$\therefore$ $f(t,y)$ satisfies the Lipscnitz condition with Lipscnitz constant $L=2$. 1

Also, $f(t,y)$ is continuous on

$D = \{(t,y) \mid 0 \le t \le 1, \; -\infty < y < \infty\}$ and 1

D is convex. 1

Hence the given IVP has a unique solution.

Q#7 [8 points] Use Euler's method to approximate the value of $y(0.9)$ using $h = 0.3$. Given

$$3\frac{dy}{dt} + 5y^2 = sin\,t, \qquad y(0) = 5.$$

(Hint: Rewrite the given ordinary differential equation.)

2

Solution: IVP: $\frac{dy}{dt} = \frac{1}{3}(\sin t - 5y^2)$, $y(0) = 5$, $h = 0.3$.

$$\therefore\ t_0 = a = 0$$
$$t_1 = a + h = 0 + 0.3 = 0.3$$
$$t_2 = a + 2h = 0 + 2(0.3) = 0.6$$
$$t_3 = a + 3h = 0 + 3(0.3) = 0.9$$

1

We want the approximate value to the solution of the given IVP.

Euler's method:

$$w_0 = w(t_0) = w(0) = 5$$

$$w_{i+1} = w_i + \frac{0.3}{3}(\sin t_i - 5w_i^2), \text{ for each } i = 0,1,2,3.$$

1

Thus,

$$w_1 = 5 + 0.1(\sin(0) - 5(5)^2) = -7.5 \approx y(0.3)$$ 1

$$w_2 = -7.5 + 0.1(\sin(0.3) - 5(-7.5)^2) = -35.5954 \approx y(0.6)$$ 1

$$w_3 = -35.5954 + 0.1(\sin(0.6) - 5(-35.5954)^2)$$ 1

$$= -669.0552$$

$$w_3 \approx y(0.9) = -669.0552$$ 1