King Fahd University of Petroleum & Minerals
Department of Mathematics
Math 371
Final Exam (Term 211)

Time: 150 minutes Key **Marks: 90**

Name: ______________________ ID#: ______________

Instructor: ______________________ Sec #: ________ Serial #: ________

- Answers should be neat, clear, and legible.
- Show all steps.
- Cell Phones and any Electronic Devices are NOT allowed.
- Use of Calculator is allowed.

Question #	Marks	Maximum Marks
1		10
2		10
3		10
4		10
5		12
6		10
7		10
8		8
9		10
Total		90

Q#1 [10 points] Use secant method $p_n = p_{n-1} - \dfrac{f(p_{n-1})(p_{n-1}-p_{n-2})}{f(p_{n-1})-f(p_{n-2})}$ to find solution of

$$sin(x) = e^{-x}$$

accurate to 10^{-2} using $p_0 = 0.5$ and $p_1 = 1.0$ (Use 4 decimal places in calculations).

Sol:

$$f(x) = \sin x - e^{-x} = 0$$

$$P_0 = 0.5,\ P_1 = 1.0$$ (2)

$$P_2 = P_1 - \frac{f(P_1)(P_1 - P_0)}{f(P_1) - f(P_0)}$$

$$= 1 - \frac{(\sin(1) - e^{-1})(1 - 0.5)}{(\sin(1) - e^{-1}) - (\sin(0.5) - e^{-0.5})}$$ (2)

$$\Rightarrow P_2 = 0.6058$$

$$P_3 = 0.6058 - \frac{(\sin(0.6058) - e^{-0.6058})(0.6058 - 1)}{(\sin(0.6058) - e^{-0.6058}) - (\sin(1) - e^{-1})}$$ (2)

$$\Rightarrow P_3 = 0.5850,$$

Similarly,

$$P_4 = 0.5886$$ (2)

Error: $|P_4 - P_3| = 0.0036 < 10^{-2}$

Thus, $P_4 = 0.5886$ (Ans:) (2)

Q#2 [10 points] Suppose $f(0) = 1, f(0.5) = 2.5, f(1) = \beta, f(0.25) = 4$, and $f(0.75) = \alpha$.

Find α and β if the Composite Trapezoidal rule and Composite Simpson's rule give the value 1.75 for

$$\int_0^1 f(x)dx.$$

Sol: Given:

$f(0)=1$, $f(0.25)=4$, $f(0.5)=2.5$, $f(0.75)=\alpha$, $f(1)=\beta$. $h = 0.25$.

$x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$. 2

$f(x_0) = 1$, $f(x_1) = 4$, $f(x_2) = 2.5$, $f(x_3) = \alpha$, $f(x_4) = \beta$.

Composite Trapezoidal Rule:

$$1.75 = \frac{0.25}{2}\left[1 + 2(4 + 2.5 + \alpha) + \beta\right]$$ 3

$$\Rightarrow 2\alpha + \beta = 0 \quad \text{— (i)}$$

Composite Simpson's Rule:

$$1.75 = \frac{0.25}{3}\left[1 + 2(2.5) + 4(4 + \alpha) + \beta\right]$$ 3

$$\Rightarrow 4\alpha + \beta = -1 \quad \text{— (ii)}$$

Solving (i) & (ii) $\Rightarrow$ $\alpha = -\frac{1}{2}$, $\beta = 1$ 2

Q#3 [10 points] For a function f, the forward-divided difference are given by

x	$f(x)$	First divided differences	Second divided differences
0	1		
		0	
1	1		c_2
		2	
2	c_1		

Find the value of $c_1 + c_2$.

Sol.:

$$2 = \frac{c_1 - 1}{2-1} \Rightarrow c_1 = 3$$ (4)

$$c_2 = \frac{2-0}{2-0} \Rightarrow c_2 = 1$$ (4)

$$\therefore c_1 + c_2 = 4$$ (2)

Q#4 [10 points] Use the Runge-Kutta method of order four to approximate $y(0.6)$ to the following initial-value problem with $N = 3$:

$$\frac{dy}{dt} = yt - t^2, \qquad 0 \le t \le 1.8, \qquad y(0) = 1.$$

Sol: $a = 0$, $b = 1.8$, $N = 3$, $h = \frac{b-a}{N} = 0.6$ ②

$f(t, y) = yt - t^2$,

$t_0 = 0$, $t_1 = 0.6$ ②

$y(0.6) \approx w_1 = w_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$\therefore$ $K_1 = hf(t_0, w_0) = 0.6(0) = 0$ ①

$K_2 = hf(t_0 + \frac{h}{2}, w_0 + \frac{1}{2}K_1) = 0.6f(0.3, 1) = 0.1260$ ①

$K_3 = hf(0.3, 1.0630) = 0.1373$ ①

$K_4 = hf(t_1, w_0 + K_3) = 0.6f(0.6, 1.1373) = 0.6(0.6824 - 0.36)$

$\Rightarrow K_4 = 0.1934.$ ①

Thus,

$y(0.6) \approx w_1 = 1 + \frac{1}{6}(0 + 2(0.1260) + 2(0.1373) + 0.1934)$ ②

$\Rightarrow y(0.6) \approx 1.1200$

Q#5 [12 points] Find the permutation matrix P such that PA can be factored into LU factorization. That is, obtain factorization in the form $A = (P^t L)U$, where $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & -1 & 1 \\ -1 & 2 & 3 & -1 \\ 3 & -1 & -1 & 2 \end{bmatrix}$.

Sol:

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 3 & 3 & 0 \\ 0 & -4 & -1 & -1 \end{bmatrix} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 - 3R_1 \end{matrix}$$

3

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 3 \end{bmatrix} \quad \begin{matrix} R_3 + 3R_2 \\ R_4 - 4R_2 \end{matrix}$$

2

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & -3 \end{bmatrix} = U \quad R_3 \leftrightarrow R_4$$

1

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & -3 & 1 & 0 \\ 3 & 4 & 0 & 0 \end{bmatrix}$$

2

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad P^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

1

Thus,

$$(P^T L)U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ -1 & -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

1

Q#6 [10 points] Use Gaussian elimination with **partial pivoting** to solve the system

$$\begin{aligned} x - y + z &= 5 \\ 7x + 5y - z &= 8 \\ 2x + y + z &= 7 \end{aligned}$$

Sol:

$$[A \mid \underline{b}] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 7 & 5 & -1 & 8 \\ 2 & 1 & 1 & 7 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 7 & 5 & -1 & 8 \\ 1 & -1 & 1 & 5 \\ 2 & 1 & 1 & 7 \end{array}\right] \quad \text{By } R_1 \longleftrightarrow R_2 \quad (2)$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 7 & 5 & -1 & 8 \\ 0 & -\frac{12}{7} & \frac{8}{7} & \frac{27}{7} \\ 0 & \frac{3}{7} & \frac{9}{7} & \frac{33}{7} \end{array}\right] \quad \text{By } R_2 - \frac{1}{7}R_1 \quad (3)$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 7 & 5 & -1 & 8 \\ 0 & -\frac{12}{7} & \frac{8}{7} & \frac{27}{7} \\ 0 & 0 & 1 & \frac{15}{4} \end{array}\right] \quad \text{By } R_3 - \left(\frac{3/7}{-12/7}\right) R_2 \text{ or } R_3 + \frac{1}{4}R_2 \quad (3)$$

Thus,

$$7x + 5y - z = 8$$

$$-\frac{12}{7}y + \frac{8}{7}z = \frac{27}{7} \quad \text{or} \quad -12y + 8z = 27$$

$$z = \frac{15}{4},$$

$$\Rightarrow \quad y = \frac{1}{4},$$

$$x = \frac{3}{2}$$

$(x, y, z) = (3/2, 1/4, 15/4)$. (Ans:) (2)

Q#7 [10 points] Find the **first two iterations** of the Gauss–Seidel method for the following linear system using $\mathbf{x}^{(0)} = (1, 1, 1)$

$$\begin{aligned} 4x_1 + x_2 - x_3 &= 5 \\ -x_1 + 3x_2 + x_3 &= -4 \\ 2x_1 + 2x_2 + 5x_3 &= 1 \end{aligned}$$

Sol:

$x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = (1, 1, 1)$ 1

First iteration:

$x_1^{(1)} = \frac{1}{4}(-x_2^{(0)} + x_3^{(0)} + 5)$ 2

$= \frac{1}{4}(-1 + 1 + 5)$

$= 1.25$

$x_2^{(1)} = \frac{1}{3}(x_1^{(1)} - x_3^{(0)} - 4)$ 2

$= \frac{1}{3}(1.25 - 1 - 4)$

$= -1.25$

$x_3^{(1)} = \frac{1}{5}(-2x_1^{(1)} - 2x_2^{(1)} + 1)$ 2

$= \frac{1}{5}(-2(1.25) - 2(-1.25) + 1)$

$= 0.2$

Second iteration ($K=2$):

$x_1^{(2)} = 1.6125 \qquad = \frac{1}{4}(-(-1.25) + 0.2 + 5)$ 1

$x_2^{(2)} = -0.8625 \qquad = \frac{1}{3}(1.6125 - 0.2 - 4)$ 1

$x_3^{(2)} = -0.1000 \qquad = \frac{1}{5}(-2(1.6125) - 2(-0.8625) + 1)$ 1

Q#8 [8 points] Use the finite difference method to approximate the solution of the boundary-value problem

$$y'' - 4y' = 4x, \qquad y(0) = 0, \;\; y(1) = 0, \text{ with } h = \frac{1}{4}.$$

Write the system $Aw = b$ (DO NOT Solve the system).

Sol: $y'' = 4y' + 0y + 4x$
where $P(x) = 4$, $V(x) = 0$, $r(x) = 4x$ (2)

$w_0 = 1$, $w_{N+1} = 0$, $a = 0$, $b = 1$.
$h = \frac{1}{4}$, where $h = \frac{b-a}{N+1} \Rightarrow \boxed{N = 3}$. (2)

Now,

$$A = \begin{bmatrix} 2+0 & -\frac{1}{2} & 0 \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ 0 & -\frac{3}{2} & 2 \end{bmatrix}_{3\times 3}$$ (2)

$$\underline{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix},$$ (1)

$$\underline{b} = \begin{bmatrix} -\left(\frac{1}{4}\right)^2 4 \cdot \frac{1}{4} + 0 \\ -\left(\frac{1}{4}\right)^2 \cdot 4 \cdot \frac{1}{2} \\ -\left(\frac{1}{4}\right)^2 \cdot 4 \cdot \frac{3}{4} + 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{16} \\ -\frac{1}{8} \\ -\frac{3}{16} \end{bmatrix}.$$ (2)

Here:
$x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{2}{4} = \frac{1}{2}$, $x_3 = \frac{3}{4}$, $x_4 = x_{N+1} = 1$. (1)

$A\underline{w} = \underline{b}$, where A, $\underline{w}$ and $\underline{b}$ is given above!

Q#9 [10 points] Write a MATLAB code using the following Algorithm (Euler's method) to solve the initial-value problem

$$y' + 2y = te^{3t}, \qquad 0 \le t \le 1, \qquad y(0)=0 \quad \text{with } h = 0.1.$$

Algorithm 5.1: EULER'S METHOD

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

at $(N+1)$ equally spaced numbers in the interval $[a, b]$:

INPUT endpoints a, b; integer N; initial condition α.

OUTPUT approximation w to y at the $(N+1)$ values of t.

Step 1 Set $h = (b-a)/N$; $t = a$; $w = \alpha$;
OUTPUT (t, w).
Step 2 For $i = 1, 2, \ldots, N$ do Steps 3, 4.
Step 3 Set $w = w + hf(t, w)$; (*Compute* w_i.)
$t = a + ih$. (*Compute* t_i.)
Step 4 OUTPUT (t, w).
Step 5 STOP.

$N = \frac{b-a}{h}$.

Sol:

```
clc
clear all
% INPUT
a = 0; b = 1; alpha = 0; h=0.1; N= b-a/h;        (2)
f =@(t,y) -2*y + t*exp(3*t);                      (2)
% Step 1
t(1)=a;
w(1)= alpha;                                      (2)
% Step 2
for i = 2 : N+1
   w(i) = w(i-1) + h*f(t(i-1), w(i-1));  % Step3, 4   (3)
   t(i) = a + (i-1)*h;
end
% OUTPUT
t
w                                                 (1)
```

Formula Sheet for Final Exam

Three-Point Endpoint Formula:

$$f'(x_0) = \frac{1}{2h}[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)] + \frac{h^2}{3}f^{(3)}(\xi_0).$$

Three-Point Midpoint Formula:

$$f'(x_0) = \frac{1}{2h}[f(x_0+h) - f(x_0-h)] - \frac{h^2}{6}f^{(3)}(\xi_1).$$

Composite Trapezoidal Rule:

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12}h^2 f''(\mu).$$

Composite Simpson's Rule:

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right] - \frac{b-a}{180}h^4 f^{(4)}(\mu).$$

Euler's method

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + hf(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1.$$

Jacobi's Method

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1\\ j\neq i}}^{n}\left(-a_{ij}x_j^{(k-1)}\right) + b_i\right], \qquad \text{for } i = 1, 2, \ldots, n.$$

The Gauss-Seidel Method

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1}(a_{ij}x_j^{(k)}) - \sum_{j=i+1}^{n}(a_{ij}x_j^{(k-1)}) + b_i\right],$$

Trapezoidal Rule:

$$\int_a^b f(x)\,dx = \frac{h}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12}f''(\xi).$$

Simpson's Rule:

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90}f^{(4)}(\xi).$$

Runge-Kutta Order Four

$$w_0 = \alpha,$$
$$k_1 = hf(t_i, w_i),$$
$$k_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right),$$
$$k_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right),$$
$$k_4 = hf(t_{i+1}, w_i + k_3),$$
$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

for each $i = 0, 1, \ldots, N-1$

Linear Finite-Difference Method

$$y'' = p(x)\,y' + q(x)y + r(x),$$
$$y(a) = \alpha, y(b) = \beta$$

$$A = \begin{bmatrix} 2+h^2q(x_1) & -1+\frac{h}{2}p(x_1) & 0 & \cdots & \cdots & 0 \\ -1-\frac{h}{2}p(x_2) & 2+h^2q(x_2) & -1+\frac{h}{2}p(x_2) & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & -1+\frac{h}{2}p(x_{N-1}) \\ 0 & \cdots & \cdots & 0 & -1-\frac{h}{2}p(x_N) & 2+h^2q(x_N) \end{bmatrix},$$

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-1} \\ w_N \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} -h^2r(x_1) + \left(1+\frac{h}{2}p(x_1)\right)w_0 \\ -h^2r(x_2) \\ \vdots \\ -h^2r(x_{N-1}) \\ -h^2r(x_N) + \left(1-\frac{h}{2}p(x_N)\right)w_{N+1} \end{bmatrix}.$$