King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 371
Exam I, Second Semester (212), 2021-2022
Net Time Allowed: 100 minutes
February 19, 2022. 1:00pm-2:40 pm

Name:____
ID #: ___________________ Section #:________

Q #	Marks	Maximum Marks
1		9
2		5
3		10
4		9
5		11
6		8
7		8

1. Write clearly.
2. Show all your steps.
3. No credit will be given to wrong steps.
4. Do not do messy work.
5. Mobile phones is NOT allowed in this exam.
6. Turn off your mobile.
7. Set your calculator to RADIAN
8. Use 4 decimal places in your calculations.

1. Let $f(x) = x^2 + \sin x$ and $x_0 = 0$, then

a) Determine the second Taylor polynomial and the reminder term for f about $x_0 = 0$. $f' = 2x + \cos x$, $f'' = 2 - \sin x$, $f''' = -\cos x$

$f(0) = 0$, $f'(0) = 1$, $f''(0) = 2$,

For $n = 2$, $x_0 = 0$;

$$x^2 + \sin x = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(\xi(x))x^3}{3!}$$

$$= x + x^2 - \frac{1}{6}\cos(\xi(x))$$

$P(x) = x + x^2$.

b) Use part(a) to approximate $\int_0^{0.1} f(x)dx$ and find the actual error in this interval.

$$I = \int_0^{0.1} f(x)dx \approx \int_0^{0.1} x + x^2 dx = 0.0053; \rightarrow \int_0^{0.1} p(x)\,dx$$

$$\text{True value of } I = \int_0^{0.1} x^2 + \sin x \, dx \approx 0.00532\,9168$$

Actual Error $= 4.165 \times 10^{-6}$

c) Use part(a) to find the bound error in this interval for $\int_0^{0.1} f(x)dx$.

$$\text{Bound error } \left|\frac{1}{6}\int_0^{0.1} x^3 \cos(\xi(x))dx\right| \leq \frac{1}{6}\int_0^{0.1} x^3 |\cos(\xi(x))|\,dx$$

$$\leq \frac{1}{6}\int_0^{0.1} x^3 dx$$

$$= 4.1\bar{6} \times 10^{-6}$$

2. Suppose p^* must approximate $p = 80$ with relative error at most 10^{-2}. Find the largest interval in which p^* must lie for each value of p.

Relative error $\left| \frac{p^* - p}{p} \right| \leq 10^{-2}$

$|p^* - 80| \leq 10^{-2} \times 80 = 0.8$

$p^* \in [79.2, 80.8]$

3. Consider the function $f(x) = e^{-x} - \sqrt{x}$.

a) With $p_0 = 0.5$ use Newtons Method to find p_2.

$f' = -e^{-x} - \frac{1}{2\sqrt{x}}$

$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$; $P_1 = 0.4234$

$P_2 = 0.4263$

b) With $p_0 = 1$ and $p_1 = 0.5$ use the Secant method to find p_3.

$P_{n+1} = P_n - \frac{f(P_n)(P_n - P_{n-1})}{f(P_n) - f(P_{n-1})}$

$P_2 = 0.4054$

$P_3 = 0.4271$

4. Given that

$$f(x) = x^2 - 4x + 4 - \ln x = 0, \quad for \;\; 2 \le x \le 4,$$

use the Bisection method to find

a) p_1, p_2, and p_3.

$f(2) = -0.69 < 0$, $f(4) = 2.6 > 0$

$P_1 = \frac{2+4}{2} = 3$; $f(3) = -0.09 < 0$

$P_2 = 3.5$; $f(3.5) = 0.99 > 0$

$P_3 = 3.25$

b) Number of iterations (n) necessarily to solve $f(x) = 0$ with accuracy 10^{-5},

$\frac{b-a}{2^n} \le 10^{-5} \Rightarrow \frac{4-2}{2^n} \le 10^{-5}$

$\Rightarrow 2^{1-n} \le 10^{-5}$ take $\log_{10}$

$\Rightarrow (1-n)\log_{10} 2 \le -5 \log_{10} 10$

$\Rightarrow 1-n \le \frac{-5}{\log_{10} 2}$

$\Rightarrow n \ge 17.6091$

$n = 18$

5. Consider $f(x) = x(1 + \ln x)$ and the nodes $x_0 = 1, x_1 = 2$, and $x_2 = 3$:

a) Use Newton's divided differences to construct the polynomial $P_2(x)$ which interpolates $f(x)$ at x_0, x_1, x_2.

$$P_2(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$$

where $a_0 = f[x_0] = \ln(1)+1 = 1$

$$a_1 = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{2(\ln 2 + 1) - (\ln 1 + 1)}{1} = 2.3863$$

$a_2 = f[\cancel{x_0}, x_1, x_2]$; $\quad f[X_1, X_2] = 2.9095$

$$= \frac{f[X_1, X_2] - f[X_0, X_1]}{X_2 - X_1} = 0.2616$$

$$P_2(x) = 1 + 2.3863(X-1) + 0.2616(X-1)(X-2)$$

b) Use the interpolating polynomial $P_2(x)$ in (a), to approximate $f(1.5)$.

$$f(1.5) \approx P_2(1.5) = 2.1278$$

6. Use Lagrange interpolating polynomial of degree three ($P_3(x)$) for the data

$$(0,0), (0.5, y_1), (1,3), (2,2)$$

to find the value of y_1 given that the coefficient of x^3 in $P_3(x)$ is 6.

$P_3(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2) + L_3(x) f(x_3)$

$x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 2$, $f(x_0) = 0$, $f(x_1) = y_1$, $f(x_2) = 3$, $f(x_3) = 2$

$$L_0(x) = \frac{(x-0.5)(x^2-3x+2)}{-1}$$

$$L_1(x) = \frac{x(x^2-3x+2)}{3/8}$$

$$L_2(x) = \frac{x(x^2-2.5x+1)}{-1/2}$$

$$L_3(x) = \frac{x(x^2-\frac{3}{2}x+\frac{1}{2})}{3}$$

$$P_3(x) = 0 + \frac{8}{3} y_1 (x^3 - 3x^2 + 2x)$$
$$- 6(x^3 - \tfrac{5}{2}x^2 + x) + \tfrac{2}{3}(x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x)$$

The Coefficient of x^3 is

$$\frac{8}{3} y_1 - 6 + \frac{2}{3} = 6$$

$$\Rightarrow y_1 = 4.25$$

7. a) Show that $\sqrt[3]{5}$ is a fixed point of the function $g(x) = x - c\frac{x^3-5}{x^2}$, where c is a fixed constant.

$$g(\sqrt[3]{5}) = \sqrt[3]{5} - c\,\frac{(\sqrt[3]{5})^3 - 5}{(\sqrt[3]{5})^2}$$

$$= \sqrt[3]{5} - c\,\frac{5-5}{(\sqrt[3]{5})^2} \stackrel{?}{=} \sqrt[3]{5}$$

b) Determine the possible values for c to ensure convergence using the fixed point $\sqrt[3]{5}$ of

$$p_{n+1} = p_n - c\frac{p_n^3 - 5}{p_n^2}, (the\ fixed-point\ iterations)$$

Note that $g'(x) = 1 - c - \frac{10c}{x^3}$

and $g'(\sqrt[3]{5}) = 1 - 3c$

then $|g'| < 1$

$\Rightarrow |1-3c| < 1$

$\Rightarrow 0 < c < 2/3$