1. Construct the natural cubic spline for the data $(1,2)$, $(2,4)$, $(3,5)$

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$n=2,\ h_0=1,\ h_1=1,\ a_0=2,\ a_1=4,\ a_2=5$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2-a_1) - \frac{3}{h_0}(a_1-a_0) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}$$

$\Rightarrow\ c_0 = 0;\ c_2 = 0;\ 4c_1 + c_2 = -3 \Rightarrow c_1 = -3/4$

$$d_j = \frac{c_{j+1} - c_j}{3h_j} \qquad d_0 = -1/4 \qquad d_1 = 1/4$$

$$b_j = \frac{(a_{j+1} - a_j)}{h_j} - h_j\,(c_{j+1} + 2c_j)/3$$

$$b_0 = 2.25 \qquad b_1 = 1.5$$

$$S(x) = \begin{cases} S_0(x) = 2 + 2.25(x-1) - \frac{1}{4}(x-1)^3 \\ S_1(x) = 4 + 1.5(x-2) - \frac{3}{4}(x-2)^2 + \cancel{1.5}\,(x-2)^3 \end{cases}$$

$\frac{1}{4}$

2. Find the least squares line approximating for the data below:

x_i	0	1	2	3	4
y_i	1.1	1.6	2.3	2.7	3.1

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Show x_i^2 and x_iy_i in a table format then solve the system of linear equations.

i	x_i	y_i	x_i^2	$x_i y_i$
1	0	1.1	0	0
2	1	1.6	1	1.6
3	2	2.3	4	4.6
4	3	2.7	9	8.1
5	4	3.1	16	12.4
Sum	10	10.8	30	26.7

$$a_0 m + a_1 \sum x_i = \sum y_i$$

$$a_0 \sum x_i + a_1 \sum x_i^2 = \sum x_i y_i$$

$$5a_0 + 10a_1 = 10.8$$

$$10a_0 + 30a_1 = 26.7$$

$$\Rightarrow \quad 5a_0 + 10a_1 = 10.8$$

$$10a_1 = 5.1$$

$$a_1 = 0.51$$

$$a_0 = 1.14$$

$$P(x_i) = 1.14 + 0.51x$$

3. In a circuit with impressed voltage $E(t)$ and inductance L, Kirchoff's first law gives the relationship

$$E(t) = L\frac{di}{dt} + R\,i(t)$$

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where R is the resistance in the circuit and i is the current. Suppose we measure the current for several values of t and obtain

t	1.00	1.01	1.02	1.03	1.04
i	3.10	3.12	3.14	3.18	3.24

where t is measured in seconds, i is in amperes, the inductance L is a constant 0.98 henries, and the resistance is 0.142 ohms. Approximate the voltage $E(t)$ when t = 1.02.

Using endpoint formula with $h = 0.01$

$t = 1.02$

$$\frac{di}{dt} \approx \frac{1}{2h}\left[-3i(1.02) + 4\,i(1.03) - i(1.04)\right]$$

$$\approx \frac{1}{0.02}\left[-3(3.14) + 4(3.18) - 3.24\right]$$

$$\approx \frac{0.06}{0.02} = 3$$

$$E(1.02) \approx 0.98(3) + (0.142)(3.14)$$

$$\approx 3.38588$$

4. The trapezoidal rule applied to $\int_0^3 f(x)dx$ gives the value 4 and Simpson's rule gives the value of 2. What is $f(1.5)$?

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$$T.R. = \int_0^3 f(x)dx \approx \frac{h}{2}\left[f(0) + f(3)\right] = 4$$

$$h = 3$$

$$f(0) + f(3) = 4 \cdot \frac{2}{3} = \frac{8}{3}$$

$$S.R. = \int_0^3 f(x)dx \approx \frac{h}{3}\left[f(0) + 4f(1.4) + f(3)\right]$$

$$h = 3/2$$

$$\frac{3}{2} \cdot \frac{1}{3}\left(f(0) + 4f(1.5) + f(3)\right) = 2$$

$$f(0) + f(3) = 8/3$$

$$\Rightarrow \frac{1}{2}\left(\frac{8}{3} + 4f(1.5)\right) = 2$$

$$\Rightarrow f(1.5) = \frac{1}{4}\left(4 - \frac{8}{3}\right) = \frac{1}{3}$$

5. Determine the values of n and h required to approximate $\int_1^2 \frac{e^x}{x} dx$ to within 10^{-5} using the composite trapezoidal rule.
(Hint: $|\frac{e^x(x^2-2x+2)}{x^3}| \leq e$)

$$\left| \frac{b-a}{12} h^2 f''(\xi) \right| \leq 10^{-5}$$

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$$h = \frac{b-a}{n} \Rightarrow n = \frac{1}{h}$$

$$\frac{1}{12} \frac{1}{n^2} |f''(\xi)| \leq 10^{-5}$$

$$f' = \frac{e^x(x-1)}{x^2}$$

$$f'' = \frac{e^x(x^2-2x+2)}{x^3}$$

from Hint

$$|f''(\xi)| \leq e$$

$$\frac{1}{12} \frac{1}{n^2} e \leq 10^{-5}$$

$$\frac{1}{n^2} \leq 0.000044$$

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$$n \geq 150.25 \qquad n = \cancel{152}$$

$$h = \frac{1}{152}$$

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6. Show that the initial value problem

$$\begin{cases} e^{t^2}y' + y = tan^{-1}y, & 0 \le t \le 2 \\ y(0) = 1, \end{cases}$$

has a unique solution.

$$f(t,y) = y' = [\tan^{-1} y - y]\, e^{-t^2}$$

f is cont.

for fixed $t \in [0,2]$

$$\frac{f(t,y_2) - f(t,y_1)}{y_2 - y_1} = \frac{\partial f}{\partial y} f(t,y)$$

$$= e^{-t^2}\left(\frac{1}{1+y^2} - 1\right)$$

$$= \frac{-y^2}{e^{t^2}(1+y^2)}$$

$$e^{-t^2} \le 1$$

$$\left|\frac{y^2}{1+y^2}\right| \le 1 \quad \forall y$$

$$|f(t,y_2) - f(t,y_1)| \le 1|y_2 - y_1|$$

$L = 1$

$\Rightarrow$ IVP has a Unique Solution

7. Consider the initial value problem

$$\begin{cases} y' - y = -t + 1 & t \in [0,1], \\ y(0) = 1, \end{cases}$$

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a) Use the Euler method with step size $h = 0.25$ to compute the approximate solution at the mesh Points.

b) The exact solution of the problem is $y(x) = e^t + t$, use the approximation obtained in part (a) to compute the actual error in approximating the solution at $t = 1$.

$f(t,y) = y' = y - t + 1$

The mesh points $\overset{i}{x_i} = \overset{0}{0}, \overset{1}{0.25}, \overset{2}{0.5}, \overset{3}{0.75}, \overset{4}{1}$

Euler, $w_0 = y_0 \quad w_{i+1} = w_i + h\, f(t_i, w_i)$

$i = 0, 1, 2, 3, 4$

$i = 0 \quad w_1 = 1 + 0.25(1 - 0 + 1) = 1.5$

$i = 1 \quad w_2 = 1.5 + 0.25(1.5 - 0.25 + 1) = 2.0625$

$i = 2 \quad w_3 = 2.0625 + 0.25(2.0625 - 0.5 + 1) = 2.7031$

$i = 3 \quad w_4 = 2.7031 + 0.25(2.7031 - 0.75 + 1) = 3.4414$

(b) The error $y(t_i) - w_i$

$y(1) = e^1 + 1 = 3.718281$

Error $= 3.718281 - 3.4414 = 0.2767$

8. Use the Runge-Kutta method of order four to approximate the solution of the I.V.P.

$$y' = f(t,y) = e^{t-y}, \ 0 \le t \le 1, \ y(0) = 1, \quad with \quad h = 0.5.$$

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$w_0 = 1, \quad t_0 = 0, \quad t_1 = 0.5, \quad t_2 = 1$

$i = 0$

$k_1 = 0.5\, e^{0-1} = 0.1839, \quad k_2 = 0.5\, e^{0.25-1.092} = 0.2154$

$k_3 = 0.5\, e^{0.25-1.1077} = 0.2121$

$k_4 = 0.5\, e^{0.5-1.2121} = 0.2453$

$\Rightarrow w_1 = 1.2140$

$i = 1$

$k_1 = 0.2448$

$k_2 = 0.2781$

$k_3 = 0.2736$

$k_4 = 0.3071$

$w_2 = 1.4899$