King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 371
Final Exam, Second Semester (212), 2021-2022
Net Time Allowed: 120 minutes
May 15, 2022. 7:00 pm-9:00 pm Bulding 54

Name:___________________________________
ID #: _________________ Section #:_______

Q #	Marks	Maximum Marks
1		8
2		8
3		5
4		5+4=9
5		5+4=9
6		9
7		9
8		9
9		4+4=8
10		4+4=8
11		8

1. Write clearly.
2. Show all your steps.
3. No credit will be given to wrong steps.
4. Do not do messy work.
5. Mobile phones is NOT allowed in this exam.
6. Turn off your mobile.
7. Set your calculator to RADIAN
8. Use 4 decimal places in your calculations.

1. Determine the coefficients so that

$$S(x) = \begin{cases} x^2 + x^3, & 0 \le x \le 1 \\ a + bx + cx^2 + dx^3, & 1 \le x \le 2 \end{cases}$$

is a cubic spline and has the property $S'''(x) = 12, \quad for \quad x > 1.$

Sol.

we have

1 $S' = \begin{cases} 2x + 3x^2 \\ b + 2cx + 3dx^2 \end{cases}$

1 $S'' = \begin{cases} 2 + 6x \\ 2c + 6dx \end{cases}$

$S''' = \begin{cases} 6 \\ 6d \end{cases} \Rightarrow 6d = 12 \Rightarrow \boxed{d = 2}$ 1.5

$\boxed{c = -2}$ 1.5

$\boxed{b = 3}$ 1.5

$\boxed{a = -1}$ 1.5

time $\approx$ 10 min

Difficulty $\approx C^+$

2. Use the gaussian elimination to solve the system and determine whether row interchanges are necessary.

$$\begin{aligned} x_2 - 2x_3 &= 4 \\ x_1 - x_2 + x_3 &= 6 \\ x_1 - x_3 &= 2. \end{aligned}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & -2 & 4 \\ 1 & -1 & 1 & 6 \\ 1 & 0 & -1 & 2 \end{array}\right] \quad 1$$

$$R_1 \leftrightarrow R_2 \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 1 & -2 & 4 \\ 1 & 0 & -1 & 2 \end{array}\right] \quad 3$$

$$-R_1 + R_2 \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 1 & -2 & 4 \\ 0 & 1 & -2 & -4 \end{array}\right] \quad 2$$

no solution , 1

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 8 \end{array}\right] \quad 1$$

3. Use three - digit chopping to calculate

$$\frac{7\pi - 22}{e - 2.70}$$

Ans:

$7\pi \simeq 7(3.14) \simeq 21.9 \Rightarrow 7\pi - 22 \simeq -0.1$

$e \simeq 2.71 \Rightarrow e - 2.70 \simeq 0.01$

$\Rightarrow \frac{7\pi - 22}{e - 2.70} \cong -10$

Time: 4 min

level: C.

4. Suppose a least square approximant $P(x) = 1.29x + a_0$ is used to fit the following data

x	1	2	3	4	5
y	1.3	3.5	4.2	w	7

a) Find w and a_0.

b) Use $P(x)$ to approximant $y(3.5)$.

Solution

x	y	x_i^2	$x_i y_i$
1	1.3	1	1.3
2	3.5	4	7.0
3	4.2	9	12.6
4	w	16	$4w$
5	7	25	35
$\sum$ 15	$16+w$	55	$55.9+4w$

2

we know $a_1 = 1.29$ and

$$a_1 = \frac{m\sum_{i=1}^{m} x_i y_i - \sum_{i=1}^{m} x_i \sum_{i=1}^{m} y_i}{m\left(\sum_{i=1}^{m} x_i^2\right) - \left(\sum_{i=1}^{m} x_i\right)^2}$$

$$1.29 = \frac{5(55.9+4w) - 15(16+w)}{5(55) - (15)^2}$$

$$\Rightarrow 1.29 = \frac{39.5+5w}{50}$$

$$\Rightarrow \boxed{w = 5}$$

2

So $\sum y_i = 21$ $\sum x_i y_i = 75.9$, so we have

$$a_0 = \frac{(55)(21) - (75.9)(15)}{5(55) - (15)^2} = \frac{16.5}{50} = 0.33$$

1

$$\Rightarrow \boxed{a_0 = 0.33}$$

) $y(3.5) \approx P(3.5) = 1.29(3.5) + 0.33 =$

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$\Rightarrow y(3.5) \approx 4.845.$

5. a) Find the first two iterations of Gauss-Seidel method to solve the following linear system

$$\begin{aligned} 12x_1 + 3x_2 - 5x_3 &= 1 \\ x_1 + 5x_2 + 3x_3 &= 28 \\ 3x_1 + 7\cancel{2}x_2 + 13x_3 &= 76, \end{aligned}$$

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starting with $[x_1 \;\; x_2 \;\; x_3] = [1 \;\; 0 \;\; 1]$.

b) What is the error $|x^{(k)} - x^{(k-1)}|$ after two iterations?

Solution

$$x_1^{(k)} = \frac{1}{12}\left(1 - 3x_2^{(k-1)} + 5x_3^{(k-1)}\right)$$

$$x_2^{(k)} = \frac{1}{5}\left(28 - x_1^{(k)} - 3x_3^{(k-1)}\right) \quad 2$$

$$x_3^{(k)} = \frac{1}{13}\left(76 - 3x_1^{(k)} - 7\cancel{2}x_2^{(k)}\right)$$

where $K = 1, 2$. $(x_1^{(0)} \;\; x_2^{(0)} \;\; x_3^{(0)}) = (1 \;\; 0 \;\; 1)$

First iteration:

1.5

$$x_1^{(1)} = \frac{1}{12}\left(1 - 3x_2^{(0)} + 5x_3^{(0)}\right) = \frac{1}{12}(1 - 0 + 5) = \frac{6}{12} = 0.5000$$

$$x_2^{(1)} = \frac{1}{5}\left(28 - 0.5 - 3(1)\right) = 4.9000$$

$$x_3^{(1)} = \frac{1}{13}\left(76 - 3(0.5) - 7\cancel{2}(4.9)\right) = \cancel{3.0923} \; -21.4077$$

Second iteration:

$\frac{1}{12}(1 - 3(4.9) + 5(-21.4077)) = -10.0615$

$\frac{1}{5}(28 + 10.0615 - 3(-21.4077)) = 20.4569$

1.5

$x_1^{(2)} = \cancel{0.14679}$

$x_2^{(2)} = \cancel{3.7153}$ -105.1317

$x_3^{(2)} = \cancel{3.8118}$ 10.5615

15.5569

Absolute Errors:

Exact: [1 3 4].

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$|x_1^{(2)} - x_1^{(1)}| = \cancel{0.35321}$ 83.7240

$|x_2^{(2)} - x_2^{(1)}| = \cancel{1.1847}$

$|x_3^{(2)} - x_3^{(1)}| = \cancel{0.4195}$

max Error 83.7240.

6. Solve the following linear system by LU – factorization

$$\begin{aligned} x_1 + x_2 - x_3 &= 4 \\ x_1 - 2x_2 + 3x_3 &= -6 \\ 2x_1 + 3x_2 + x_3 &= 7 \end{aligned}$$

Solution $A\underline{x} = \underline{b} \Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$

$\because A = LU$

$\therefore LU\underline{x} = \underline{b}$ — (i)

Let $\underline{y} = U\underline{x}$ — (ii) $\Rightarrow L\underline{y} = \underline{b}$ — (iii) $\Rightarrow \underline{y} \to \underline{x}$ by (ii)

$A \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 1 & 3 \end{bmatrix}$ $R_2 - R_1$ ($m_{21} = 1$), $R_3 - 2R_1$ ($m_{31} = 2$) $\quad L = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix}$

$\to \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & \frac{13}{3} \end{bmatrix}}_{U}$ 2 $\quad R_3 + \frac{1}{3}R_2$ ($m_{32} = -\frac{1}{3}$) $\quad \frac{4}{3} + 3 = \frac{13}{3}$ $\quad L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -\frac{1}{3} & 1 \end{bmatrix}$ 2

Now, (iii) $\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -\frac{1}{3} & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$

$\Rightarrow y_1 = 4, \quad y_1 + y_2 = -6, \quad 2y_1 - \frac{1}{3}y_2 + y_3 = 7$ 2

$\Rightarrow y_1 = 4, \; y_2 = -10, \; y_3 = -\frac{13}{3}$

Now, (ii) $\Rightarrow U\underline{x} = \underline{y} \Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & \frac{13}{3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ -\frac{13}{3} \end{bmatrix}$ 3

$\Rightarrow \boxed{x_1 = +1, \; x_2 = 2, \; x_3 = -1}$

7. Set up the linear system of equations $Aw = b$ to solve the boundary–value problem

$$y'' = -3y' + 2y + 2x + 3, \quad 0 \le x \le 1$$

$$y(0) = 2, \quad y(1) = 1,$$

using the linear finite difference method with $h = 1/4$. Do not solve the system.

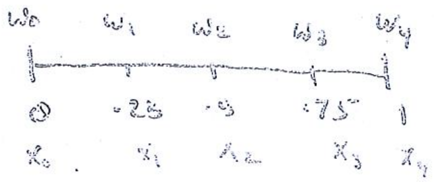
Sol.

Note the following:

$P(x) = -3$, $q(x) = 2$, $r(x) = 2x+3$ 2

$h = \frac{b-a}{N+1} \Rightarrow N+1 = \frac{1.0}{1/4}$

$\Rightarrow N+1 = 4 \Rightarrow \boxed{N=3}$



2

3

$$A = \begin{bmatrix} \frac{17}{8} & -\frac{11}{8} & 0 \\ -\frac{5}{8} & \frac{17}{8} & -\frac{11}{8} \\ 0 & -\frac{5}{8} & \frac{17}{8} \end{bmatrix}, \quad b = \begin{bmatrix} \frac{33}{32} \\ -\frac{1}{4} \\ \frac{17}{16} \end{bmatrix}$$ 2

The Solution $\begin{bmatrix} 0.8495 \\ 0.5628 \\ 0.6655 \end{bmatrix}$

Note that in the formula sheet the matrix A and R.H.S b is given.

* time required ≈ 12 min
* Difficulty level ≈ C+

8. Use Gaussian elimination with partial pivoting to solve the linear system with backward substitution and two–digit rounding arithmetic

$$\begin{aligned} x_1 + 4x_2 + 2x_3 &= 11 \\ 2x_1 + 2x_2 + 5x_3 &= 3 \\ 4x_1 + x_2 - x_3 &= 8. \end{aligned}$$

Time: 9 min

Level: C+

$$x_1 + 4x_2 + 2x_3 = 11$$
$$2x_1 + 2x_2 + 5x_3 = 3$$
$$4x_1 + x_2 - x_3 = 8$$

Ans:

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 11 \\ 2 & 2 & 5 & 3 \\ 4 & 1 & -1 & 8 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 4 & 1 & -1 & 8 \\ 2 & 2 & 5 & 3 \\ 1 & 4 & 2 & 11 \end{array}\right]$$

$$\xrightarrow[-0.25R_1+R_3]{-0.50R_1+R_2} \left[\begin{array}{ccc|c} 4 & 1 & -1 & 8 \\ 0 & 1.5 & 5.5 & -1 \\ 0 & 3.8 & 2.3 & 9 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 4 & 1 & -1 & 8 \\ 0 & 3.8 & 2.3 & 9 \\ 0 & 1.5 & 5.5 & -1 \end{array}\right] \xrightarrow{-\frac{1.5}{3.8}R_2+R_3} \left[\begin{array}{ccc|c} 4 & 1 & -1 & 8 \\ 0 & 3.8 & 2.3 & 9 \\ 0 & 0 & 4.6 & -4.5 \end{array}\right]$$

$\to 0.39$

$$x_3 = \frac{-4.5}{4.6} = -0.98$$

$$3.8x_2 = 0.98(2.3) + 9 = 11 \quad \Rightarrow \quad x_2 = 2.9$$

$$4x_1 = -2.9 + (-0.98) + 8 = 4.1$$

$$\Rightarrow \quad x_1 = \frac{4.1}{4} = 1.0$$

1 2 1 2 3

9. Given the linear system

$$4x_1 - 12\alpha x_2 = 6$$

$$6\alpha x_1 - 2x_2 = 3.$$

a) Find the value(s) of α for which the system has no solution.

b) Find the value(s) of α for which the system has an infinite number of solution.

The Augmented system is

$$\begin{array}{c} 1.5\alpha \rightarrow \\ (-1) \rightarrow \end{array} \begin{pmatrix} 4 & -12\alpha & 6 \\ 6\alpha & -2 & 3 \end{pmatrix} \quad 2$$

$$\Rightarrow \begin{pmatrix} 6\alpha & -18\alpha^2 & 9\alpha \\ -6\alpha & +2 & -3 \end{pmatrix} \quad 1$$

$$\Rightarrow \begin{pmatrix} 6\alpha & -18\alpha^2 & 9\alpha \\ & 2-18\alpha^2 & 9\alpha-3 \end{pmatrix} \quad 1$$

when $2 - 18\alpha^2 = 0$ and $9\alpha - 3 = 0$

$\Downarrow$

$\alpha^2 = \frac{1}{9}$ $\qquad \alpha = \frac{1}{3}$

$\alpha = \pm\frac{1}{3}$

2 If $\alpha = \frac{1}{3}$ both side equal $= 0$ (b) infinite Solns

2 If $\alpha = -\frac{1}{3}$ only coff of x_2 = equal $= 0$ (a) no Soluti.

10. Let $g(x) = \sqrt{2x+8}$.

a) If $p_0 = 5$, find p_1, p_2 using fixed-point method.

b) Show that $g(x)$ has unique fixed point in the interval $[3, 5]$.

a) $P_1 = g(5) = \sqrt{10+8} = 4.2426$ 2

$P_2 = g(\sqrt{18}) = \sqrt{2\sqrt{18}+8} = 4.0602$ 2

b) $g \in [3,5]$ $\quad g'(x) = \dfrac{2}{2\sqrt{2x+8}}$ 1

$|g'(x)| = \left|\dfrac{1}{\sqrt{2X+8}}\right| \leq 0.2670$ 2

$\forall x \in [3,5]$

1 $0.267 \in (0,1)$

$\Rightarrow$ $g(x)$ conv to the unique f.P in $[3,5]$

11. Water flows from an inverted conical tank with circular orifice(opening) at the rate

$$\frac{dy}{dt} = -0.6\pi r^2\sqrt{-2g}\frac{\sqrt{y}}{A(y)},$$

where r is the radius of the orifice, y is the height of the liquid level from the vertex of the cone, and $A(y)$ is the area of the cross section of the tank y units above the orifice. Suppose $r = 0.2m$, $g = -9.8m/s^2$, and $A(y) = 0.14\pi y^2$. The tank has an initial water level of $8m$. Use the Euler's method to find the following:

a) The water level after $60s$ with $h = 20$ s.

b) When the tank will be empty, with $h = 20$ s.

$$\frac{dy}{dt} = -0.6\,\pi\,(0.2)^2\sqrt{19.6}\cdot\frac{\sqrt{y}}{0.14\,\pi\,y^2}$$

$$= -0.7589\,y^{-\frac{3}{2}}$$

$y_0 = 8$

$w_0 = 8;\quad w_{i+1} = w_i + 20\left(-0.7589\,w_i^{-\frac{3}{2}}\right)$

$w_1 = 7.3292$ 20s

4

$w_2 = 6.5643$ 40s

$w_3 = 5.6618$ 60s. (a) → 5.6618 m

$w_4 = 4.5352$ 80s

$w_5 = 2.9635$ 100s

4

$w_6 = -0.0113$ 2m

The tank will be empty in 2min.