King Fahd University of Petroleum and Minerals

Department of Mathematics

Math 371 Exam 1, 1st Semester (231),

Net Time Allowed: 90 minutes

October 7, 2023

Name: KEY

ID No.:

Section NO.:

Please:

1. Write clearly with a **pen or dark pencil** in the **designed area for each question.**

2. **Fill your info clearly**, and write your **ID NO** in the pages (3, 5, 7, 9) in the right corner **inside the box.**

3**. If you need more space**, you may use page 9 and 10 but you have to state that clearly in the question's area.

4. Show **all** your steps, no credit will be given to wrong steps.

5. Set your calculator to RADIAN

Q1. (Q13, sec. 1.1)

(a) Find the third Taylor polynomial $P_3(x)$ for the function $f(x) = (x-1)\ln x$ about $x_0 = 1$.

(b) Find an upper bound for the error when $\int_{0.5}^{1.5} P_3(x)dx$ is used to approximate $\int_{0.5}^{1.5} f(x)dx$.

Sol: Third Taylor polynomial is

$$P_3(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \frac{(x-x_0)^3}{3!}f'''(x_0)$$

$$f'(x) = \ln x + \frac{x-1}{x} = \ln x + 1 - \frac{1}{x}, \qquad f''(x) = \frac{1}{x} + \frac{1}{x^2}, \qquad f'''(x) = -\frac{1}{x^2} - \frac{2}{x^3}$$

$$f(1) = 0, \qquad f'(1) = 0, \qquad f''(1) = 2, \qquad f'''(1) = -3$$

$$P_3(x) = 0 + 0 + (x-1)^2 - \frac{1}{2}(x-1)^3 = (x-1)^2 - \frac{1}{2}(x-1)^3$$

(b)

$$\left| \int_{0.5}^{1.5} f(x)dx - \int_{0.5}^{1.5} P_3(x)dx \right| \le \int_{0.5}^{1.5} |f(x) - P_3(x)|dx$$

$$\le \max_{0.5\le\xi\le1.5} \int_{0.5}^{1.5} |R_3(x)|\, dx = \max_{0.5\le\xi\le1.5} \frac{1}{24}\left(\frac{2}{\xi^3} + \frac{6}{\xi^4}\right) \int_{0.5}^{1.5} (x-1)^4\, dx$$

$$= \frac{1}{24}(16+96)\left.\frac{(x-1)^5}{5}\right|_{0.5}^{1.5} = \frac{112}{24}\left(\frac{1}{80}\right) = 0.0583$$

Q2: (Q14, sec. 1.2)

Let

$$f(x) = \frac{e^x - e^{-x}}{x}$$

(a) Use three-digit rounding arithmetic to evaluate $f\ (0.1)$.

(b) If the actual value is $f\ (0.1)\ =\ 2.003335$. Find the relative error for the values obtained in parts (a).

Sol: (a)

$$e^{0.1} = 1.1051709 = 1.11,\ \ e^{-0.1} = 0.9048374 = 0.905$$

$$f(0.1) = \frac{1.11 - 0.905}{0.100} = 2.05$$

(b)

The actual value is,

$$f(0.1) = 2.003335$$

The absolute error is,

$$|2.003335 - 2.05| = 0.0467$$

The relative error is,

$$\frac{0.0467}{2.003335} = 0.00166.$$

Q3. (Q18, sec. 2.1)

(a) Find a bound for the number of iterations needed by the Bisection method to achieve an approximation with accuracy 0.1 to the solution of $x^3 + x - 4 = 0$ lying in the interval $[1, 2]$.

(b) Find an approximation to the root with this degree of accuracy.

Sol: (a) Using the theorem 2.1,

$$|p_N - p| \leq \frac{b_1 - a_1}{2^N} = \frac{2-1}{2^N} = \frac{1}{2^N} < 0.1$$

$$2^N > 10 \ \text{ or } N > \frac{\ln 10}{\ln 2} = 3.3219$$

Take $N = 4$.

(b) Approximate the root of $f(x) = x^3 + x - 4 = 0$ on the interval $[1, 2]$ using Bisection method.

Set $a_1 = 1,\ b_1 = 2$ and calculate $f(a_1) = f(1) = -2 < 0,\ f(b_1) = f(2) = 6 > 0$

$$\frac{(b_1 - a_1)}{2} = \frac{(1-0)}{2} = 0.5 \not< 0.1$$

Since $f(a_1) \cdot f(b_1) < 0$, there is a root in $[a_1,\ b_1]$.

Find $p_1 = \frac{a_1+b_1}{2} = \frac{1+2}{2} = 1.5$ and calculate $f(p_1) = f(1.5) = 0.8750 > 0$

Set $a_2 = a_1 = 1,\ b_2 = p_1 = 1.5$ and find $p_2 = \frac{a_2+b_2}{2} = \frac{1+1.5}{2} = \frac{2.5}{2} = 1.25$

$$\frac{(b_2 - a_2)}{2} = \frac{(1.5-1)}{2} = 0.25 \not< 0.1$$

Calculate $f(p_2) = -0.7969 < 0$. Note $f(a_2) = f(1) = -2,\ f(b_2) = 0.8750$

Set $a_3 = p_2 = 1.25,\ b_3 = b_2 = 1.5$, find $p_3 = \frac{a_3+b_3}{2} = \frac{1.25+1.5}{2} = \frac{2.75}{2} = 1.3750.$

$$\frac{(b_3 - a_3)}{2} = \frac{(1.5-1.25)}{2} = 0.125 \not< 0.1$$

Calculate $f(p_3) = -0.0254 < 0$. Note $f(a_3) = -0.7969 < 0,\ f(b_3) = 0.8750 > 0$.

Set $a_4 = p_3 = 1.3750,\ b_4 = b_3 = 1.50$, find $p_4 = \frac{a_4+b_4}{2} = \frac{1.375+1.5}{2} = 1.4375$.

$\frac{(b_4-a_4)}{2} = \frac{(1.5-1.375)}{2} = 0.0625 < 0.1$, STOP.

Q4. (Q7, sec. 2.2)

Use a **fixed-point** iteration method to determine a solution accurate to within 10^{-2} for $x^4 - 3x^2 - 3 = 0$ on [1, 2]. Use $p_0 = 1$ and $g(x) = (3x^2 + 3)^{\frac{1}{4}}$.

Sol:

$$p_1 = g(p_0) = g(1) = 1.5651$$

$$p_2 = g(p_1) = g(1.5651) = 1.7936,$$

$$|p_2 - p_1| = |1.7936 - 1.5651| = 0.2285 \nless 10^{-2}$$

$$p_3 = g(p_2) = g(1.7936) = 1.8859$$

$$|p_3 - p_2| = |1.8859 - 1.7936| = 0.0923 \nless 10^{-2}$$

$$p_4 = g(p_3) = g(1.8859) = 1.9228$$

$$|p_4 - p_3| = |1.9228 - 1.8859| = 0.0369 \nless 10^{-2}$$

$$p_5 = g(p_4) = g(1.9228) = 1.9375$$

$$|p_5 - p_4| = |1.9375 - 1.9228| = 0.0147 \nless 10^{-2}$$

$$p_6 = g(p_5) = g(1.9375) = 1.9433$$

$$|p_6 - p_5| = |1.9433 - 1.9375| = 0.0058 < 10^{-2}$$

Solution is 1.9433.

STOP.

Q5. (~Q7, sec. 2.3)

Use the **secant method** to an approximation to the solution of $\ln(x-1)+\cos(x-1)=0$. Let $p_0=1.3$ and use the Newton's method to find the second initial. Apply two iterations of the secant method.

Sol: Let $f(x)=\ln(x-1)+\cos(x-1)$, then, $f'(x)=\frac{1}{x-1}-\sin(x-1)$

Newton's Method

$$p_1=p_0-\frac{f(p_0)}{f'(p_0)}=1.3-\frac{\ln(1.3-1)+\cos(1.3-1)}{\frac{1}{1.3-1}-\sin(1.3-1)}=1.3818$$

Now Secant Method

Let $p_0=1.3$, $p_1=1.3818$, then,

$$p_2=p_1-\frac{f(p_1)(p_1-p_0)}{f(p_1)-f(p_0)}$$

$$=p_1-\frac{(\ln(p_1-1)+\cos(p_1-1))(p_1-p_0)}{(\ln(p_1-1)+\cos(p_1-1))-(\ln(p_0-1)+\cos(p_0-1))}=1.3951$$

$$p_3=p_2-\frac{f(p_2)(p_2-p_1)}{f(p_2)-f(p_1)}$$

$$=p_2-\frac{(\ln(p_2-1)+\cos(p_2-1))(p_2-p_1)}{(\ln(p_2-1)+\cos(p_2-1))-(\ln(p_1-1)+\cos(p_1-1))}=1.3977$$

Q6. (~Q13, sec. 3.1)

(a) Construct the **Lagrange interpolating** polynomial for the function $f(x) = e^x$ on $[-1, 1]$ using the nodes $x_0 = -1,\ x_1 = 0,\ x_2 = 1.$

(b) Find a bound for the absolute error on the interval $[-1\ 1]$.

Sol: (a) $f(x_0) = f(-1) = 0.3679, \quad f(x_1) = f(0) = 1, \quad f(x_2) = f(1) = 2.7183$

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 0)(x - 1)}{(-1 - 0)(-1 - 1)} = \frac{1}{2}(x^2 - x)$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x + 1)(x - 1)}{(0 + 1)(0 - 1)} = -(x^2 - 1)$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x + 1)(x - 0)}{(1 + 1)(1 - 0)} = \frac{1}{2}(x^2 + x)$$

So,

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$= \frac{1}{2}(x^2 - x)(0.3679) - (x^2 - 1)(1) + \frac{1}{2}(x^2 + x)(2.7183)$$

$$= 0.54308x^2 + 1.1752x + 1$$

(b) By the **Theorem 3.3**

$$f(x) = e^x, \qquad f'(x) = e^x, \quad f''(x) = e^x, \quad f'''(x) = e^x$$

$$E_2(x) = \frac{f'''(\xi(x))}{3!}(x - x_0)(x - x_1)(x - x_2) = \frac{e^\xi}{3!}(x + 1)(x - 0)(x - 1) = \frac{e^\xi}{3!}(x^3 - x)$$

Let $g(x) = x^3 - x, then\ \ g'(x) = 3x^2 - 1 = 0 \implies x = \pm\frac{1}{\sqrt{3}}$, these are the critical points

$$g(-1) = 0, \quad g\left(\frac{-1}{\sqrt{3}}\right) = 0.3849, \qquad g\left(\frac{1}{\sqrt{3}}\right) = -0.3849, \quad g(1) = 0$$

Absolute Maximim of $g(x) = 0.3849$

$$Max|E_2(x)| \leq \frac{e}{6}(0.3849) = 0.1744$$

Q7. (Q7, sec. 3.3)

(a) Use Divided -difference method to construct the interpolating polynomial of a heghest degree for the points given in the following table:

x	-0.1	0.0	0.2	0.3
$f(x)$	5.3	2.0	3.19	1.0

(b) Add $f(0.35) = 0.9726$ to the lable and construct the interpolating polynomial of a higher degree.

Sol:

-0.1	5.3				
0.0	2.0	$\frac{2-5.3}{0+0.1} = -33$			
0.2	3.19	$\frac{3.19-2.0}{0.2-0.0} = 5.95$	$\frac{5.95+33}{0.2+0.1} = 129.83$		
0.3	1.0	$\frac{1-3.19}{0.3-0.2} = -21.9$	$\frac{-21.9-5.95}{0.3-0.0} = -92.83$	$\frac{-92.83-129.83}{0.3+0.1} = -556.67$	
0.35	0.9726	$\frac{0.9726-1}{0.35-0.3} = -0.548$	$\frac{-0.548+21.9}{0.35-0.2} = 142.35$	$\frac{142.35+92.83}{0.35-0.0} = 671.94$	$\frac{671.94+556.66}{0.35+0.1} = 2730.24$

$$a_0 = 5.3, \quad a_1 = -33, \quad a_2 = 129.83, \quad a_3 = -556.67$$

$$P_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$$= 5.3 - 33(x + 0.1) + 129.83(x + 0.1)(x) - 556.67(x + 0.1)(x)(x - 0.2)$$

$$P_4(x) = P_3(x) + 2730.24(x + 0.1)(x)(x - 0.2)(x - 0.3).$$