# King Fahd University of Petroleum and Minerals

## Department of Mathematics

## Math 371 Exam 2, 1st Semester (231),

## Net Time Allowed: 90 minutes

## November 14, 2023

**Name:** Key **(Capital Letters)**

**ID No.:**

**Section NO.:**

Please:

1. Write clearly with a **pen or dark pencil** in the **designed area for each question.**
2. **Fill your info clearly**, and write your **ID NO** in the pages (3, 5, 7, 9) in the right corner **inside the box.**
3. **If you need more space**, you may use page 9 and 10 but you have to state that clearly in the question's area.
4. Show **all** your steps, no credit will be given to wrong steps.
5. Set your calculator to RADIAN and Use **4 decimal places for all calculations.**

**Q1)** Consider the following initial-value problem: 14 points

$$\frac{dy}{dt} - 1 = \frac{y}{t}, \qquad 1 \le t \le 2, \qquad y(1) = 2,$$

a. Use the **Euler's method** with step size $h = 0.25$ to approximate $y(2)$. (Use 4 decimal places for all calculations.)

b. If the exact solution is $y(t) = t \ln t + 2t$, find an **error bound** for the approximation of $y(2)$ in part a.

**Sol: (a)** Here $f(t,y) = 1 + \frac{y}{t}$, $h = 0.25$, $t_0 = 1$, $t_1 = 1.25, t_2 = 1.5\,, t_3 = 1.75, t_4 = 2$,

$$w_0 = \alpha = 2$$

$$w_{i+1} = w_i + hf(t_i, w_i)$$

$$w_1 = w_0 + hf(t_0, w_0) = w_0 + h\left(1 + \frac{w_0}{t_0}\right) = 2 + 0.25\left(1 + \frac{2}{1}\right) = 2.75$$

$$w_2 = w_1 + hf(t_1, w_1) = w_1 + h\left(1 + \frac{w_1}{t_1}\right) = 2.75 + 0.25\left(1 + \frac{2.75}{1.25}\right) = 3.55$$

$$w_3 = w_2 + hf(t_2, w_2) = w_2 + h\left(1 + \frac{w_2}{t_2}\right) = 3.55 + 0.25\left(1 + \frac{3.55}{1.5}\right) = 4.3917$$

$$w_4 = w_3 + hf(t_3, w_3) = w_3 + h\left(1 + \frac{w_3}{t_3}\right) = 4.3917 + 0.25\left(1 + \frac{4.3917}{1.75}\right) = 5.2691$$

$$y(2) \approx 5.2691.$$

**(b)** $y(t) = t \ln t + 2t \quad \Longrightarrow \quad y'(t) = \ln t + 1 + 2, \quad y''(t) = \frac{1}{t}$

$$M = \max_{1 \le t \le 2} |y''(t)| = 1, \qquad \frac{\partial f}{\partial y} = \frac{1}{t} \quad \Longrightarrow L = 1$$

**Error Bound** $$|y(2) - w_4| \le \frac{hM}{2L}\left(e^{L(t_4 - 1)} - 1\right) = \frac{0.25 * 1}{2 * 1}\left(e^{1*(2-1)} - 1\right) = 0.2148$$

ID NO:

**Q2)** Use the **Composite Simpson's rule** to approximate the integral $\int_{-0.5}^{0.5} x\ln(x+1)\ dx$ with $n=6$.

10 points

**Sol:** $h=\frac{0.5-(-0.5)}{6}=\frac{1}{6}$

$$x_0=-\frac{1}{2},\ x_1=-\frac{1}{2}+\frac{1}{6}=-\frac{1}{3},\quad x_2=-\frac{1}{3}+\frac{1}{6}=-\frac{1}{6},\quad x_3=-\frac{1}{6}+\frac{1}{6}=0$$

$$x_4=0+\frac{1}{6}=\frac{1}{6},\quad x_5=\frac{1}{6}+\frac{1}{6}=\frac{1}{3},\quad x_6=\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$$

$$\int_{-0.5}^{0.5} x\ln(x+1)\ dx\ =\frac{\frac{1}{6}}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+4f(x_5)+f(x_6)]$$

$$=\frac{1}{18}[0.3466+4(0.1352)+2(0.0304)+4(0)+2(0.0257)+4(0.0959)+0.2027]$$

$$=0.0881$$

**Q3)** Show that the initial-value problem

$$\frac{dy}{dt} = \frac{2}{t}y + t^2e^t, \qquad 1 \le t \le 2, \qquad y(1) = 0,$$

10 points

has a **unique solution on** $D = \{(t, y)|1 \le t \le 2, -\infty < y < \infty\}$. (Do not solve the IVP)

**Sol:** Here $f(t, y) = \frac{2}{t}y + t^2e^t$ is **continuous** on $D$.

$$\frac{\partial f}{\partial y} = \frac{2}{t} \quad and \quad \left|\frac{\partial f}{\partial y}\right| \le 2 = L \quad \text{on } 1 \le t \le 2$$

$f(t, y)$ satisfy the **Lipschitz condition** on $D$ in the variable $y$.

By Theorem 5.4, the IVP has a unique solution.

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**Q4)** If the **least squares polynomial** of degree one for the following table

12 points

| $x_i$ | 1.0 | 1.1 | 1.3 | 1.5 | 1.9 | 2.1 |
|---|---|---|---|---|---|---|
| $y_i$ | $\alpha$ | 1.96 | 2.21 | 2.45 | 2.94 | $\beta$ |

is $y = 0.6209 + 1.2196x$, find the values of $\alpha$ and $\beta$.

**Sol:** Here $a = 0.6209$, $b = 1.2196$

| $x_i$ | $y_i$ | $x_i^2$ | $x_i\, y_i$ |
|---|---|---|---|
| 1.0 | $\alpha$ | 1.0 | $\alpha$ |
| 1.1 | 1.96 | 1.21 | 2.1560 |
| 1.3 | 2.21 | 1.69 | 2.8730 |
| 1.5 | 2.45 | 2.25 | 3.675 |
| 1.9 | 2.94 | 3.61 | 5.5860 |
| 2.1 | $\beta$ | 4.41 | $2.1\beta$ |
| **8.9** | $\boldsymbol{9.56 + \alpha + \beta}$ | **14.17** | $\boldsymbol{\alpha + 2.1\beta + 14.29}$ |

Put in the equations,

$$na + \left(\sum x\right) b = \sum y$$
$$\left(\sum x\right) a + \left(\sum x^2\right) b = \sum x\, y$$

$$6(0.6209) + 8.9(1.2196) = 9.56 + \alpha + \beta$$
$$8.9\,(0.6209) + 14.17(1.2196) = \alpha + 2.1\beta + 14.29$$

$$\alpha + \beta = 5.0198$$

$$\alpha + 2.1\,\beta = 8.5177$$

$$\beta = \frac{8.5177 - 5.0198}{1.1} = 3.1799, \quad \alpha = 5.0198 - 3.1799 = 1.8399$$

**Q5)** The **Trapezoidal rule** applied to $\int_0^{1/2} f(x)\,dx$ gives the value 4 and **Simpson's rule gives** the value 1. What is $f\left(\frac{1}{4}\right)$?

8 points

**Sol:** By Trapezoidal rule

$$\int_0^{\frac{1}{2}} f(x)dx = \frac{\frac{1}{2}}{2}\left[f(0) + f\left(\frac{1}{2}\right)\right] = 4 \quad \Longrightarrow \quad f(0) + f\left(\frac{1}{2}\right) = 16$$

By the Simpson's rule

$$\int_0^{\frac{1}{2}} f(x)dx = \frac{\frac{1}{4}}{3}\left[f(0) + 4f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right)\right] = 1 \quad \Longrightarrow \quad f(0) + 4f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) = 12$$

$$4f\left(\frac{1}{4}\right) = 12 - 16 = -4$$

$$f\left(\frac{1}{4}\right) = -1.$$

ID NO:

**Q6)** Consider the following table:

| $x$ | $f(x)$ | $f'(x)$ |
|---|---|---|
| $-0.3$ | $-0.27652$ | |
| $-0.2$ | $-0.25074$ | |
| $-0.1$ | $-0.16134$ | |

14 points

a) Use the most accurate **three-point formula** to determine each missing entry in the table.

b) Given that $f'(0) \approx 1.9731$, approximate $f''(-0.1)$?

**Sol:** (a) Here $h = 0.1$

$$f'(-0.3) \approx \frac{-3f(-0.3) + 4f(-0.3+0.1) - f(-0.3+0.2)}{2(0.1)} = \frac{-3(-0.27652) + 4(-0.25074) - (-0.16134)}{0.2}$$
$$= -0.06030$$

$$f'(-0.2) \approx \frac{f(-0.2+0.1) - f(-0.2-0.1)}{2(0.1)} = \frac{-0.16134 + 0.27652}{0.2} = 0.57590$$

$$f'(-0.1) \approx \frac{-3f(-0.1) + 4f(-0.2) - f(-0.3)}{2(-0.1)} = \frac{-3(-0.16134) + 4(-0.25074) - (-0.27652)}{-0.2} = 1.21210$$

(b)

$$f''(-0.1) = \frac{f(-0.1-0.1) - 2f(-0.1) + f(-0.1+0.1)}{(0.1)^2} = \frac{-0.25074 - 2(-0.16134) - f(0)}{0.01}$$

Now, using $f'(0) \approx 1.9731$

$$f'(0) = \frac{-3f(0) + 4f(-0.1) - f(-0.2)}{2(-0.1)} = \frac{-3f(0) + 4(-0.16134) - (-0.25074)}{-0.2} = 1.9731$$

Implies $f(0) \approx 0$.

Therefore,

$$f''(-0.1) \approx \frac{-0.25074 - 2(-0.16134)}{0.01} = 7.1940.$$

Or, $f'(0) = \frac{f(0-0.1) - f(0)}{-0.1} = \frac{-0.16134 - f(0)}{-0.1} = 1.9731, \textit{implies } f(0) \approx -0.0360.$

Therefore, $f''(-0.1) \approx \frac{-0.25074 - 2(-0.16134) + 0.0360}{0.01} = 10.7940$. Or any similar idea.

**Q7)** Construct the **Natural cubic spline** $S(x)$ that passes through the points $(0,1), (\frac{\pi}{2}, 2)$, and $(\pi, 1)$.

12 points

**Sol:** $x_0 = 0, \quad x_1 = \frac{\pi}{2}, \quad x_2 = \pi$

$a_0 = f(x_0) = 1, \quad a_1 = f(x_1) = 2, \quad a_2 = f(x_2) = 1$

The two polynomials are:

$$S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3$$
$$= a_0 + b_0 x + c_0 x^2 + d_0 x^3$$

and

$$S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3$$
$$= a_1 + b_1\left(x - \frac{\pi}{2}\right) + c_1\left(x - \frac{\pi}{2}\right)^2 + d_1\left(x - \frac{\pi}{2}\right)^3$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{\pi}{2} & 2\pi & \frac{\pi}{2} \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ -\frac{12}{\pi} \\ 0 \end{bmatrix}$$

$$c_0 = 0, \quad c_2 = 0, \quad c_1 = -\frac{6}{\pi^2}$$

$$b_0 = \frac{a_1 - a_0}{h_0} - \frac{h_0(c_1 + 2c_0)}{3} = \frac{1}{\frac{\pi}{2}} - \frac{\frac{\pi}{2}\left(-\frac{6}{\pi^2}\right)}{3} = \frac{2}{\pi} + \frac{1}{\pi} = \frac{3}{\pi}$$

$$b_1 = \frac{a_2 - a_1}{h_1} - \frac{h_1(c_2 + 2c_1)}{3} = -\frac{1}{\frac{\pi}{2}} - \frac{\frac{\pi}{2}\left(-\frac{12}{\pi^2}\right)}{3} = -\frac{2}{\pi} + \frac{2}{\pi} = 0$$

$$d_0 = \frac{(c_1 - c_0)}{3h_0} = \frac{-\frac{6}{\pi^2}}{3\left(\frac{\pi}{2}\right)} = -\frac{4}{\pi^3}$$

$$d_1 = \frac{(c_2 - c_1)}{3h_1} = \frac{\frac{6}{\pi^2}}{3\left(\frac{\pi}{2}\right)} = \frac{4}{\pi^3}$$

$$S_0(x) = 1 + \frac{3}{\pi}x - \frac{4}{\pi^3}x^3$$

$$S_1(x) = 2 - \frac{6}{\pi^2}\left(x - \frac{\pi}{2}\right)^2 + \frac{4}{\pi^3}\left(x - \frac{\pi}{2}\right)^3$$

$$S(x) = \begin{cases} 1 + \frac{3}{\pi}x - \frac{4}{\pi^3}x^3, & x \in \left(0, \frac{\pi}{2}\right) \\ 2 - \frac{6}{\pi^2}\left(x - \frac{\pi}{2}\right)^2 + \frac{4}{\pi^3}\left(x - \frac{\pi}{2}\right)^3, & x \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$