Solution Math371-241-05 Exam2

1. The Euler method is used to approximate the solution of the initial value problem

$$y' = 1 + \frac{y}{t}, \quad y(1) = 2, \quad 1 \le t \le 2,$$

with $h = 0.25$. If $y(t) = t\ln(t) + 2t$ is the exact solution, then the error bound $|y(t_2) - w(t_2)|$ is less or equal to,
Note: this is not the absolute error.

(a) 0.0811
(b) 0.0112
(c) 0.0545
(d) 0.0699
(e) 0.0775

Sol:

$$\frac{\partial f}{\partial y} = \frac{1}{t}, \quad L = \max_{1\le t\le 2}\left\{\frac{1}{t}\right\} = 1, \qquad y' = \ln t\ + 1 + 2, \quad y'' = \frac{1}{t}, \quad M = \max_{1\le t\le 2}\left\{\frac{1}{t}\right\} = 1$$

$$|y(t_2) - w(t_2)| \le \frac{hM}{2L}\left(e^{L(t_2-1)} - 1\right) = \frac{0.25}{2}\left(e^{0.5} - 1\ \right) = 0.0811$$

2. If the Runge-Kutta method of order 4 is used to approximate the solution of the initial value problem

$$y' = \cos(2t) + \sin(3t), \quad y(0) = 1, \quad 0 \le t \le 1,$$

with $h = 0.25$, then $w(t_1)$ is equal to,

(a) 1.3292
(b) 1.2735
(c) 1.2179
(d) 2.9750
(e) 1.6537

Sol: $w_0 = 1,\ t_0 = 0, t_1 = 1.25,\ f(t,y) = \cos(2t) + \sin(3t)$

$$k_1 = hf(t_0, w_0) = 0.25(\cos(0) + \sin(0)) = 0.25$$

$$k_2 = hf\left(t_0 + \frac{h}{2}, w_0 + \frac{k_1}{2}\right) = 0.25\left(\cos(0.25) + \sin\left(\frac{3(0.25)}{2}\right)\right) = 0.3338$$

$$k_3 = hf\left(t_0 + \frac{h}{2}, w_0 + \frac{k_2}{2}\right) = 0.25\left(\cos(0.25) + \sin\left(\frac{3(0.25)}{2}\right)\right) = 0.3338$$

$$k_4 = hf(t_1, w_0 + k_3) = 0.25(\cos(0.25) + \sin\left(\frac{3(0.25)}{2}\right) = 0.3898$$

$$w_1 = w_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 + \frac{1}{6}(0.25 + 2(0.3338) + 2(0.3338) + 0.3898) = 1.3292$$

3. The linear system

$$\begin{aligned} 2x_1 + 6\alpha x_2 &= 4, \\ \alpha x_1 + 3x_2 &= 6, \end{aligned}$$

has a unique solution for any value of α **except**,

(a) $\alpha = \pm 1$
(b) $\alpha = 1$
(c) $\alpha = -1$
(d) $\alpha = 3$
(e) $\alpha = \pm\frac{1}{3}$

Sol: $\left(\begin{matrix} 2 & 6\alpha \\ \alpha & 3 \end{matrix}\middle|\begin{matrix} 4 \\ 6 \end{matrix}\right) \quad \begin{matrix} \\ -\frac{\alpha}{2}R_1 + R_2 \end{matrix} \left(\begin{matrix} 2 & 6\alpha \\ 0 & -3\alpha^2 + 3 \end{matrix}\middle|\begin{matrix} 4 \\ -2\alpha + 6 \end{matrix}\right) \longrightarrow \left(\begin{matrix} 2 & 6\alpha \\ 0 & -3(\alpha^2 - 1) \end{matrix}\middle|\begin{matrix} 4 \\ -2(\alpha - 3) \end{matrix}\right)$

The system has a unique solution if $\alpha \neq \pm 1$.

If $\alpha = 1$, then $0x_2 = 4$, no solution.

If $\alpha = -1$, then $0x_2 = 8$, no solution.

4. The row interchanges required to solve the linear system,

$$\begin{aligned} 5x_1 + x_2 - 6x_3 &= 7 \\ 2x_1 + x_2 - x_3 &= 8 \\ 6x_1 + 12x_2 + x_3 &= 9 \end{aligned}$$

using partial pivoting are,

(a) Interchange rows 1 and 3, then rows 2 and 3
(b) Interchange rows 1 and 2, then rows 2 and 3
(c) Interchange rows 1 and 3 only
(d) Interchange rows 2 and 3 only
(e) No row interchange required for partial pivoting

Sol: $\left(\begin{matrix} 5 & 1 & -6 \\ 2 & 1 & -1 \\ 6 & 12 & 1 \end{matrix}\middle|\begin{matrix} 7 \\ 8 \\ 9 \end{matrix}\right)$ $\max\{5, 2, 6\} = 6$, in R_3, Interchange $R_1 \leftrightarrow R_3$. $\left(\begin{matrix} 6 & 12 & 1 \\ 2 & 1 & -1 \\ 5 & 1 & -6 \end{matrix}\middle|\begin{matrix} 9 \\ 8 \\ 7 \end{matrix}\right)$

$\begin{matrix} -\frac{1}{3}R_1 + R_2 \\ -\frac{5}{6}R_1 + R_3 \end{matrix} \left(\begin{matrix} 6 & 12 & 1 \\ 0 & -3 & -\frac{4}{3} \\ 0 & -9 & -\frac{41}{6} \end{matrix}\middle|\begin{matrix} 9 \\ 5 \\ -\frac{1}{2} \end{matrix}\right)$ $\max\{|-3|, |-9|\} = 9$, in R_3, Interchange $R_2 \leftrightarrow R_3$

5. If the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix}$ is written as LU where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_1 & 1 & 0 \\ l_2 & l_3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_1 & u_2 & u_3 \\ 0 & u_4 & u_5 \\ 0 & 0 & u_6 \end{bmatrix}$, then $l_1 + l_2 + l_3 + u_4 + u_5 + u_6$ is equal to,

(a) 12
(b) 16
(c) 6
(d) 10
(e) 14

Sol: $\begin{matrix} -\frac{3}{2}R_1 + R_2 \\ -\frac{3}{2}R_1 + R_2 \end{matrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & \frac{9}{2} & \frac{15}{2} \\ 0 & \frac{9}{2} & \frac{7}{2} \end{bmatrix} \quad -R_2 + R_3 \begin{bmatrix} 2 & -1 & 1 \\ 0 & \frac{9}{2} & \frac{15}{2} \\ 0 & 0 & -4 \end{bmatrix} = U, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ \frac{3}{2} & 1 & 1 \end{bmatrix}$

$$l_1 + l_2 + l_3 + u_4 + u_5 + u_6 = \frac{3}{2} + \frac{3}{2} + 1 + \frac{9}{2} + \frac{15}{2} - 4 = 12$$

6. Consider the linear system,

$$\begin{aligned} x_1 - x_2 &= 2 \\ 2x_1 + 2x_2 + 3x_3 &= -1 \\ -x_1 + 3x_2 + 2x_3 &= 4 \end{aligned}$$

If the coefficient matrix A is factorized as LU and $LY = b$ is solved for $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then the sum of y_1, y_2, y_3 is equal to,

(a) 5.5
(b) 8.5
(c) 2.5
(d) 6
(e) 6.5

Sol: $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 3 \\ -1 & 3 & 2 \end{bmatrix} \Rightarrow \begin{matrix} -2R_1 + R_1 \\ R_1 + R_2 \end{matrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 3 \\ 0 & 2 & 2 \end{bmatrix} \Rightarrow -\frac{1}{2}R_2 + R_3 \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = U$

$$LY = b \;\Rightarrow\; \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \quad y_1 = 2, \; y_2 = -5, \quad y_3 = \frac{17}{2}$$

$$y_1 + y_2 + y_3 = 2 - \frac{5}{2} + \frac{17}{2} = \frac{11}{2} = 5.5$$

7. If $A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 4 & 5 \\ 2 & 3 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$, then $\|A\|_1 + \|b\|_2$ is equal to,

(a) 14
(b) 16
(c) 9
(d) 10
(e) 12

Sol: $\|A\|_1 = \max\{|1| + |-2| + |2|,\ \ |2| + |4| + |3|,\ \ |1| + |5| + |1|\} = \max\{5, 9, 7\} = 9$

$$\|b\|_2 = \sqrt{4^2 + 3^2 + 0^2} = \sqrt{25} = 5$$

$$\|A\|_1 + \|b\|_2 = 9 + 5 = 14$$

8. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, then $\|A\|_2$ is equal to

(a) 5
(b) 4
(c) 6
(d) 13
(e) 25

Sol:

$A^T A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$, $\quad \begin{vmatrix} 5 - \lambda & 10 \\ 10 & 20 - \lambda \end{vmatrix} = (5 - \lambda)(20 - \lambda) - 100 = \lambda^2 - 25\lambda = 0$

$$\lambda = 0,\ \ 25$$

$$\|A\|_2 = \sqrt{25} = 5$$

9. If the following system is solve using the Jacobi method with initial guess $\mathbf{x} = [0, 0, 0]$

$$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 6 \\ x_1 - 3x_2 + x_3 &= 9 \\ 3x_1 + 2x_2 + x_3 &= 2 \end{aligned}$$

then $x_1^{(2)} + x_2^{(2)} + x_3^{(2)}$ is equal to

(a) $\frac{31}{6}$
(b) $\frac{43}{6}$
(c) $\frac{47}{6}$
(d) $\frac{59}{6}$
(e) $\frac{25}{6}$

Sol:

$$\begin{aligned} x_1^{(1)} &= \tfrac{1}{2}\left(6 - x_2^{(0)} + 3x_3^{(0)}\right) = \tfrac{1}{2}(6 - 0 + 0) = 3 \\ x_2^{(1)} &= -\tfrac{1}{3}\left(9 - x_1^{(0)} - x_3^{(0)}\right) = -\tfrac{1}{3}(9 - 0 - 0) = -3 \\ x_3^{(1)} &= 2 - 3x_1^{(0)} - 2x_2^{(0)} = 2 - 0 - 0 = 2 \end{aligned}$$

$$\begin{aligned} x_1^{(2)} &= \frac{1}{2}\left(6 - x_2^{(1)} + 3x_3^{(1)}\right) = \frac{1}{2}(6 + 3 + 6) = \frac{15}{2} \\ x_2^{(2)} &= -\frac{1}{3}\left(9 - x_1^{(1)} - x_3^{(1)}\right) = -\frac{1}{3}(9 - 3 - 2) = -\frac{4}{3} \\ x_3^{(2)} &= 2 - 3x_1^{(1)} - 2x_2^{(1)} = 2 - 9 + 6 = -1 \end{aligned}$$

$$x_1^{(2)} + x_2^{(2)} + x_3^{(2)} = \frac{15}{2} - \frac{4}{3} - 1 = \frac{31}{6}$$

10. If the following system is solve using the Gauss-Siedel method with initial guess $\mathbf{x} = [1, 2, 3]$

$$\begin{aligned} x_1 - x_2 + 3x_3 &= -2 \\ 2x_1 + x_2 - x_3 &= 3 \\ 3x_1 + 2x_2 + x_3 &= 4 \end{aligned}$$

then $x_1^{(1)} + x_2^{(1)} + x_3^{(1)}$ is equal to

(a) -2
(b) 2
(c) 3
(d) 4
(e) -3

Sol:

$$\begin{aligned} x_1^{(1)} &= -2 + x_2^{(0)} - 3x_3^{(0)} = -2 + 2 - 9 = -9 \\ x_2^{(1)} &= 3 - 2x_1^{(1)} + x_3^{(0)} = 3 + 18 + 3 = 24, \qquad -9 + 24 - 17 = -2 \\ x_3^{(1)} &= 4 - 3x_1^{(1)} - 2x_2^{(1)} = 4 + 27 - 48 = -17 \end{aligned}$$

11. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $P_2 P_1 A = \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, then the permutation matrices P_1 and P_2 are,

(a) $P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(b) $P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(c) $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(d) $P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e) $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Sol: (a)

12. If the following system is solved using the Conjugate Gradient method with two-digit rounding and $\mathbf{x}^{(0)} = [0, 0]^T$,

$$\begin{aligned} x_1 + \frac{1}{2}x_2 &= \frac{5}{6} \\ \frac{1}{2}x_1 + \frac{1}{3}x_2 &= \frac{11}{12} \end{aligned}$$

then $\mathbf{x}^{(1)}$ is equal to,

(a) $[0.69, \ \ 0.76]^T$

(b) $[0.96, \ \ 0.89]^T$

(c) $[0.54, \ \ 0.92]^T$

(d) $[0.89, \ \ 0.95]^T$

(e) $[0.50, \ \ 0.33]^T$

Sol: Let $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $\mathbf{r}^{(0)} = \mathbf{b} - A\mathbf{x}^{(0)} = \mathbf{b} = \begin{bmatrix} 0.83 \\ 0.92 \end{bmatrix}$ and $\mathbf{v}^{(1)} = \mathbf{r}^{(0)} = \begin{bmatrix} 0.83 \\ 0.92 \end{bmatrix}$

$$t_1 = \frac{\langle \mathbf{r}^{(0)}, \ \mathbf{r}^{(0)} \rangle}{\langle \mathbf{v}^{(1)}, \ A\mathbf{v}^{(1)} \rangle} = \frac{[0.83 \quad 0.92] \begin{bmatrix} 0.83 \\ 0.92 \end{bmatrix}}{[0.83 \quad 0.92] \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.33 \end{bmatrix} \begin{bmatrix} 0.83 \\ 0.92 \end{bmatrix}} = \frac{1.5}{[0.83 \quad 0.92] \begin{bmatrix} 1.3 \\ 0.72 \end{bmatrix}} = \frac{1.5}{1.8} = 0.83$$

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + t_1 \mathbf{v}^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0.83 \begin{bmatrix} 0.83 \\ 0.92 \end{bmatrix} = \begin{bmatrix} 0.69 \\ 0.76 \end{bmatrix}$$

13. If $P_1(x) = a + bx$ with $b = 0.5$ is the linear least square polynomial for the data,

$$(0,1),\ (1,0),\ (2,2),\ (3,\alpha),\ (4,2),$$

then $P_1(2.5)$ is equal to,

(a) 1.85
(b) 3.00
(c) 1.29
(d) 2.02
(e) 1.95

Sol: Here $b = 0.5$

x_i	y_i	x_i^2	$x_i y_i$
0	1	0	0
1	0	1	0
2	2	4	4
3	α	9	3α
4	2	16	8
10	5+α	30	12+3 α

$$\begin{matrix} 5a + 10b = 5 + \alpha \\ 10a + 30b = 12 + 3\alpha \end{matrix} \Rightarrow \begin{matrix} 5a + 10(0.5) = 5 + \alpha \\ 10a + 30(0.5) = 12 + 3\alpha \end{matrix} \Rightarrow \begin{matrix} 5a - \alpha = 0 \\ 10a - 3\alpha = -3 \end{matrix} \Rightarrow \alpha = 5a$$

$$10a - 15a = -3 \ \Rightarrow -5a = -3 \quad \Rightarrow \quad a = 0.6, \quad \alpha = 3.$$

$$P_1(x) = 0.6 + 0.5x, \quad P_1(2.5) = 0.6 + 0.5(2.5) = 1.85$$

14. If the vector $V = \begin{bmatrix} 3 \\ -8 \\ -1 \end{bmatrix}$ is an Eigenvectors of the matrix $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix}$, then the corresponding eigenvalue is,

(a) -2
(b) 3
(c) 4
(d) 2
(e) -3

Sol: $AV = \lambda V \ \Rightarrow \ \begin{bmatrix} 3 & 2 & -1 \\ 1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -8 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 - 16 + 1 \\ 3 + 16 - 3 \\ 6 + 0 - 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 16 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 3 \\ -8 \\ -1 \end{bmatrix}$

So, the corresponding eigenvalue is $\lambda = -2$