King Fahd University of Petroleum and Minerals
Department of Mathematics

Math 371

Final Exam

251

December 15, 2025

Net Time Allowed: 120 Minutes

MASTER VERSION

1. Consider the linear system

$$\begin{cases} 6x_1 + x_2 = 7 \\ x_1 + 3x_2 = 4 \end{cases}$$

Let $X^{(1)}$ be the first iteration of the **conjugate gradient method** with the intial guess $X^{(0)} = (0,0)^t$. Then $\| X^{(1)} \|_\infty =$

(a) 1.143 ________________ (correct)

(b) 0.653

(c) 1.000

(d) 6.000

(e) 7.000

2. Let $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$, $v_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ Which of the following is **FALSE?**

(a) v_1 is an eigenvector of the matrix A. ________________ (correct)

(b) v_1 and v_2 are A-orthogonal.

(c) A is symmetric and positive definite (SPD).

(d) v_1 and v_3 are orthogonal.

(e) $\lambda = 3$ is an eigenvalue of the matrix A.

3. Let A be a 4×3 matrix with singular value decomposition

$$A = \begin{bmatrix} 36 & -0.008 & 48 \\ 24 & 0 & -18 \\ 48 & 0.006 & 64 \\ -32 & 0 & 24 \end{bmatrix} = \begin{bmatrix} 0.6 & 0 & -0.8 & 0 \\ 0 & -0.6 & 0 & 0.8 \\ 0.8 & 0 & 0.6 & 0 \\ 0 & 0.8 & 0 & 0.6 \end{bmatrix} \begin{bmatrix} 100 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 0.01 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.6 & -0.8 & 0 \\ 0 & 0 & 1 \\ 0.8 & 0.6 & 0 \end{bmatrix}^{\mathbf{t}}.$$

Find the rank-1 (low-rank) approximation A_1 of A. Which of the following matrices is A_1? (Hint: A_1 contains the most significant singular value)

(a) $A_1 = \begin{bmatrix} 36 & 0 & 48 \\ 0 & 0 & 0 \\ 48 & 0 & 64 \\ 0 & 0 & 0 \end{bmatrix}$ ________________________________(correct)

(b) $A_1 = \begin{bmatrix} 36 & 0 & 48 \\ 24 & 0 & -18 \\ 48 & 0 & 64 \\ -32 & 0 & 24 \end{bmatrix}$

(c) $A_1 = \begin{bmatrix} 36 & 0 & 48 \\ 0 & 0 & 0 \\ 48 & 0 & 64 \\ -32 & 0 & 24 \end{bmatrix}$

(d) $A_1 = \begin{bmatrix} 36 & -0.001 & 48 \\ 24 & 0 & -18 \\ 48 & 0.001 & 64 \\ -32 & 0 & 24 \end{bmatrix}$

(e) $A_1 = \begin{bmatrix} 35.99 & 0 & 47.99 \\ 0 & 0 & 0 \\ 47.99 & 0 & 63.99 \\ 0 & 0 & 0 \end{bmatrix}$

4. Which of the following matrices are orthogonal matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) A and B ______________________________ (correct)

(b) A only

(c) C only

(d) B and C

(e) A, B and C

5. **A natural cubic spline** S is defined by

$$S(x) = \begin{cases} s_1(x) = 2 + \frac{3}{4}(x-1) + \frac{1}{4}(x-1)^3, & 1 \le x \le 2, \\ s_2(x) = 3 + \frac{3}{2}(x-2) + a(x-2)^2 - b(x-2)^3, & 2 \le x \le 3. \end{cases}$$

If S interpolates the data $(1,2), (2,3)$ and $(3,c)$, then the value of c is equal to

(a) 5 ———————————————— (correct)

(b) 4

(c) 3

(d) 2

(e) 1

6. Use the **composite Trapezoidal method** with $n = 4$ to approximate the integral $\int_0^4 \frac{x}{x^2+4} dx \approx$

(a) 0.7808 ———————————————— (correct)

(b) 0.8047

(c) 0.8048

(d) 0.8541

(e) 0.7047

7. Let $X^{(0)} = (0,0,0)^t$. If the second iteration of the **Gauss-Siedel method** for the system $\begin{cases} 4x_1 - x_2 + 2x_3 = 5, \\ -x_1 + 4x_2 - x_3 = 2 \\ 2x_1 - x_2 + 4x_3 = 5 \end{cases}$ is $X^{(2)}$. Then $\|\, X^{(2)} \,\|_\infty =$

(a) 1.039 ____(correct)

(b) 0.967

(c) 0.972

(d) 1.143

(e) 1.000

8. Let $\mathbf{x}^{(0)} = (0,2)^t$. If the first iteration of the **Steepest Descent method** for the function

$$g(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 2)^2 + 100 + x_2^4$$

is $\mathbf{x}^{(1)}$, then with the help of the MATLAB code compute $\mathbf{x}_1^{(1)} + \mathbf{x}_2^{(1)} =$

```
clear;
g = @(x) (x(1)-1).^2 + (x(2)-2).^2 + 100 + (x(2)).^4;
grad_g = @(x)[ 2*(x(1)-1) ; 2*(x(2)-2)+ 4*(x(2))^3 ];
x0 = [0; 2];
z = grad_g(x0)/norm(grad_g(x0));
alpha1 = 0;  alpha2 = 1/2; alpha3 = 1;
g1 = g(x0 - alpha1*z),
  g1 = 117
g2 = g(x0 - alpha2*z),
  g2 = 106.2633
g3 = g(x0 - alpha3*z),
  g3 = 102.8831
```

(a) 1.0833 ____(correct)

(b) 2.0074

(c) 1.1081

(d) 2.0003

(e) 1.0391

9. If **Midpoint method** is used to approximate the solutions for the initial value problem:

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5,$$

with $h = 0.2$. Then $y(0.4) \approx$

(a) 1.21136 ____(correct)
(b) 1.21408
(c) 1.20692
(d) 1.21456
(e) 1.22137

10. Use the **Gram–Schmidt process** to determine a set of orthogonal vectors $\{v_1, v_2, v_3\}$ from the linearly independent vectors

$$X_1 = (1,1,1)^t, \qquad X_2 = (0,4,2)^t, \qquad X_3 = (2,0,4)^t.$$

(a) $\{q_1 = (1,1,1)^t, q_2 = (-2,2,0)^t, q_3 = (-1,-1,2)^t\}$ ______________________(correct)
(b) $\{q_1 = (1,1,1)^t, q_2 = (0,-2,3)^t, q_3 = (4,1,2)^t\}$
(c) $\{q_1 = (1,1,1)^t, q_2 = (0,1,-1)^t, q_3 = (-5,1,1)^t\}$
(d) $\{q_1 = (1,1,1)^t, q_2 = (-2,2,0)^t, q_3 = (-2,3,3)^t\}$
(e) $\{q_1 = (1,1,1)^t, q_2 = (0,-3,3)^t, q_3 = (-2,2,2)^t\}$

11. If $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$, then the **spectral radius** of A is $\rho(A) =$

   (a) 4
   (b) 2
   (c) 8
   (d) 1
   (e) 0

12. Consider solving the linear system $Ax = b$ using **Conjugate Gradient method** with initial guess $\mathbf{x}^{(0)} = (0, 0, 0)^t$. The following MATLAB code performs the first CG iteration and compute $\mathbf{x}^{(1)}$:

```
clear
A=[2 -1 0;-1 3 -1;0 -1 2]; b = [3;1;1];  x0 =[0;0;0];
r0 = b-A*x0;  v1 = r0;
t1 = (r0'*r0)/(v1'*A*v1)
   t1 = 0.7333
x1 = x0 + t1*v1
   x1  = 3x1
          2.2000
          0.7333
          0.7333
r1 = r0 - t1*A*v1
   r1  = 3x1
          -0.6667
           1.7333
           0.2667
```

   Then determine the value of $\|\, \mathbf{x}^{(2)} \,\|_\infty =$

   (a) 2.3043 ____(correct)
   (b) 1.4638
   (c) 2.6154
   (d) 1.4315
   (e) 2.0045

13. Given the following data points: $(1, 2.0)$, $(2, 3.1)$, $(4, 7.5)$. Construct the **least squares** approximation of the form

$$P(x) = \; b\; e^{\;a\,x}$$

Then $P(3) =$

(a) 4.824 ______________________________(correct)

(b) 5.445

(c) 4.924

(d) 4.747

(e) 5.004

14. If $P_1(x) = 0.4 + a_1\; x$ is the **linear least squares** polynomial for the data points:

$$(0, 1), (1, 0), (2, 2), (3, 4), (4, b).$$

Then $P_1(5) =$

(a) 4.40 ______________________________(correct)

(b) 2.70

(c) 5.93

(d) 2.95

(e) 5.75

15. The **singular value decomposition (SVD)** of the matrix

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

is given by $A = USV^t$. To help determine U, S and V, consider the following MATLAB commands and their outputs:

```
clear
A = [2 0 2;1 0 -1;0 0 0;0 0 0;0 -1 0];
[V1,D1]= eig(A'*A)
```

```
V1 =                                    D1 =
    0.7071    0.7071      0              2     0     0
         0        -1      0              0     3     0
   -0.7071         0  0.7071             0     0     8
```

```
[V2,D2] = eig(A*A')
```

```
V2 =                                     D2 =
      0         0     0       0     1          0     0     0     0     0
      0         0     1       0     0          0     0     0     0     0
 -0.4082    0.7071    0  0.5774     0          0     0     2     0     0
 -0.4082   -0.7071    0  0.5774     0          0     0     0     3     0
 -0.8165         0    0 -0.5774     0          0     0     0     0     8
```

Using this information, compute the value of

$$U_{52} + U_{33} + V_{13} + S_{22} =$$

(a) 1.8618 ____ (correct)

(b) 4.7624

(c) 1.7625

(d) 1.0012

(e) 1.2987

16. Suppose the singular value decomposition of a matrix A is given by

$$USV^t = \begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 0.07 & 0 \\ 0 & 0.01 \end{bmatrix} \begin{bmatrix} 0.28 & 0.96 \\ -0.96 & 0.28 \end{bmatrix}^t,$$

and let $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. If $x = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$ is the solution of the linear system $Ax = b$, then $a_0 + a_1 =$

(a) 11.2 ____(correct)

(b) 24.8

(c) -13.6

(d) 12.4

(e) 15.6

17. Let $\mathbf{x}^{(0)} = (2,2)^t$. If the first iteration of **Newton's method** for the nonlinear system $\begin{cases} x_1^2 + x_2 - 1 = 0, \\ (x_1-1)^2 + (x_2-2)^2 - 3 = 0, \end{cases}$ is $\mathbf{x}^{(1)}$, then $||\, \mathbf{x}^{(1)} \,||_\infty$ =

(a) 7 ______________________________ (correct)

(b) 3

(c) 10

(d) 5

(e) 1

18. Let $\mathbf{x}^{(0)} = (0,1)^t$ and $\alpha = 0.1$. If the second iteration of the **Steepest Descent method** for the function

$$g(x_1, x_2) = (x_1^2 + 2\ x_2^2 - 4)^2 + (x_1\ x_2 - 1)^2$$

is $\mathbf{x}^{(2)}$, then $||\, \mathbf{x}^{(2)} \,||_\infty$ =

(a) 17.2656 ______________________________ (correct)

(b) 13.2025

(c) 15.5508

(d) 16.4000

(e) 12.3900

19. If the Linear Finite Difference method is used to approximate the solution of the boundary value problem

$$y'' + (1+3x)\ y' = (2+x)\ y + x, \quad 0 \le x \le 3, \quad y(0) = 1, \quad y(3) = 2,$$

with $h = 1$, then $y(1) \approx$

(a) 0.24 ____(correct)

(b) 0.39

(c) 0.44

(d) 0.47

(e) 0.27

20. Consider the linear system

$$A\ w\ =\ b$$

obtained from applying the linear finite difference method with $h = 1$ to the boundary value problem

$$y'' = 8y' + 11y + 5x, \quad 0 \le x \le 4, \quad y(0) = 2,\ y(4) = -1,$$

Compute $\quad ||\ A\ ||_\infty \ +\ ||\ b\ ||_\infty \ =$

(a) 33 ____(correct)

(b) 17

(c) 27

(d) 24

(e) 57