Math405: Learning From Data
Major Exam 2

16th November 2022 at 6:00pm[1]

[a] *Duration 120 minutes*
[b] *Plagiarism is strictly prohibited*

NAME:

KFUPM ID:

1. Changes in A^{-1} (5 points)

For the matrix:

$$A = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}.$$

Find the inverse of $M = A - u^t v$, where $u = (1\ 1)$ and $v = (1\ 1)$.

N.B.: You have to use the formula seen in class.

2. NMF

Consider the matrix

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$$

a. Starting with an initial $U_0^t = (1, 1)$, detail manually the **two** first full iterations that would solve:

$$minimize \, \|A - UV\|_F^2 \, , \text{ with } U, V \geq 0.$$

b. Find A_1, the closest rank one approximation of A.

3. Pseudoinverse A^+ (5 points)

a. Find the pseudoinverse matrix A^+ for the given matrix:

$$A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 1 & 2 \end{pmatrix}.$$

b. For $b^t = (1\ 1)$, show that $x^+ = A^+b$ is the least squares solution to $Ax = b$.

4. Generalized SVD (5 points)

Consider the two matrices A and B such that:

$$A^t = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B^t = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Find analytically the GSVD factorization matrices U_a, U_b Z, Σ_a and Σ_b of A and B, such that $A = U_a \Sigma_a Z$ and $B = U_b \Sigma_b Z$. Detail all your steps.

5. Arnoldi Iteration (5 points)

Consider the matrix A such that:

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

a. Perform manually the Arnoldi iteration with $b^t = (1, 1)$ and $n = 1$ to obtain the matrix Q and the vector h.

b. Write down the matrix H such that $H = Q^t A Q$.

c. What are from H the approximated values of the largest and the least eigenvalues of A?

6. Gram-Schmidt (5 points)

Consider the matrix A such that:

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}.$$

a. Perform manually the Gram-Schmidt factorization $A = QR$.

b. Verify that your solution satisfies the mathematical conditions:

$$A^t A = R^t R \text{ and } A^{-1} = R^{-1} Q^t.$$