Math405: Learning From Data
Final Exam

2nd January 2023 at 8:00am[a]

[a] *slimb@kfupm.edu.sa*
[b] *Duration: 120 minutes*

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1. SVD and Pseudo-Inverse (5 points)

Consider the matrix $A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$.

a) Find the Singular Vector Decomposition $A = U^t \Sigma V$.

b) Find the pseuod-inverse of A.

2. Newton-Raphson (5 points)

Consider the problem where we have to:

minimize $f(x_1, x_2) = (2x_1 - 1)^2 + (x_2 - 2)^2 + x_1 x_2$, starting from $X_0 = (-1, 1)$.

a) Perform manually one full iteration of the Newton-Raphson method to obtain the next point X_1.

b) Does your new point satisfy the optimiality conditions $\nabla f = 0$ and H positive-definite?

3. Gram-Schmidt (5 points)

Find the Gram-Schmidt factorization of the matrix $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$.

4. Derivative of A^{-1} and $\sigma(t)$ (5 points)

Consider the matrix $A(t) = \begin{pmatrix} 3t & 2t \\ 1 & 0 \end{pmatrix}$.

a) Find $\frac{\partial A^{-1}}{\partial t}$.

b) Find the derivatives $\frac{\partial \sigma_{1,2}}{\partial t}$ of the singular values σ_1 and σ_2 of A.

5. Matrix Completion (5 points)

a) Find the matrix $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with the smallest l_2 norm $||S||_2$ and such that:

$$S \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

b) Find the best possible real-valued matrix A^* that completes $A = \begin{pmatrix} * & 3 \\ -1 & * \end{pmatrix}$ and minimizes:

$$\min_{C \text{ and } R} \frac{1}{2}||(A - CR^t)_{known}||_2^2 + \frac{1}{2}(||C||_F^2 + ||R||_F^2)$$

where $CR^t = (2 \times 1)(1 \times 2)$.

6. Backpropagation (5 points)

In a simple neural network, we have two inputs p_1 and p_2, one output y, and one unique neuron in the output layer forecasting the value of y.

Assume that $\hat{y} = f(w_1 p_1 + w_2 p_2 + b) = w_1 p_1^2 + w_2 p_2^2 + w_3 p_1 + w_4 p_2 + b$.

a) Find the general expression of the error function gradient.

b) Find the general expression of the error function Hessian matrix. Is it invertible?