Name: ID#:

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1. [10pts] (a) Let G be a group of order $2p^2$, where p is prime. Show that G contains a unique Sylow p-subgroup.

(b) Let G be a simple group of order $4p^2$, where p is an odd prime. Find p.

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2. [10pts] (a) Prove that if a group has order 15, then it is cyclic.

(c) Deduce from (a) that there is no group G such that $|G/Z(G)| = 15$, where $Z(G)$ is the center of G.

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3. [10pts] (a) Let R be an integral domain and p be a prime element of R.

(i) Prove that p is irreducible.

(ii) Suppose $p|q$, where q is an irreducible element of R. Must we have $q|p$? Justify.

(b) Define what is meant by a UFD and give an example of a UFD that is not a PID.

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4. [10pts] (a) Let d be a square-free integer and let $x \in \mathbb{Z}\left[\sqrt{d}\right]$ be such that $N(x)$ is prime (where $N\left(a + b\sqrt{d}\right) = |a^2 - db^2|$ for $a, b \in \mathbb{Z}$). Prove that x is irreducible.

(b) Let $R = \mathbb{Z}\left[\sqrt{-2p}\right]$ where p is an odd prime.

(i) Prove that $\sqrt{-2p}$ is irreducible in R.

(ii) Find integers $a, b > 1$ such that $\dfrac{\left(a + \sqrt{-2p}\right)\left(b + \sqrt{-2p}\right)}{\sqrt{-2p}} \in \mathbb{Z}$, and deduce that $\sqrt{-2p}$ is not prime in R.