KFUPM

Department of Mathematics

Math 427, Final Exam, Term 242.

Duration: 2.5 hours

Part I (100 points)

1. **[10 points]** Find all integers that can be written simultaneously in the form $3m + 2$ and in the form $7k - 4$, where $m\ and\ k$ are integers.
2. **[10 points]** Find all primes p such that $p^2 | 3^{p^2} + 7$.
3. **[6+6+8=20 points]**
   a. Let n be a positive integer. Find the value of the sum $\sum_{d|n} \frac{\mu(d)\tau(d)}{\sigma(d)}$.
   b. Solve $\tau(n) = 10$. Find the smallest solution.
   c. Find all perfect numbers of the form $3p^k$, where p is a prime number and k is a positive integer.
4. **[6+4+4+6=20 points]**
   a. Show that 3 is a primitive root modulo 31.
   b. Find a reduced residue system modulo 31 consisting of powers of 3.
   c. Which powers of 3 are also primitive roots modulo 31?
   d. Solve $x^4 \equiv 9\ mod\ 31$. You may give your solutions as powers of 3.
5. **[20 points]** Find all primes p such that $\left(\frac{7}{p}\right) = 1$.
6. **[10 points]** Find all primitive Pythagorean triples (x, y, z), with y even, in which $z = 65$.
7. **[10 points]** Solve $\frac{1}{x} + \frac{1}{y} = \frac{2}{5}$ in integers.

Part II (50 points)

8. **[8+7 points]**
   a. Use numbers of the form $n! + 1$ to prove the infinitude of primes.
   b. Prove that there are infinitely many composite numbers of the form $n! + 1$.
9. **[10 points]** Let n be a positive integer. Prove that $\phi(n) \geq \frac{n}{2^{\omega(n)}}$.
10. **[5+10 points]** Let p be an odd prime, $a\ and\ h$ be positive integers.
    a. Prove that if $a^h \equiv 1\ mod\ p$, then $a^{ph} \equiv 1\ mod\ p^2$.

b. Prove that if g is a primitive root modulo p^2, then it is a primitive root modulo p. **Hint:** Use Part (a).

11.[10 points] Prove that if $p \; and \; q$ are distinct odd primes such that $p \equiv -q \; mod \; 4$, then $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$.

Good luck,

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