# King Fahd University of Petroleum and Minerals
# College of Computing and Mathematics
# Department of Mathematics

Math 432 - Major Exam I

AY 2023-2024 (Term 232)

Time Allowed: 120 Minutes

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**Name:** ........................................................................................ **ID number:** .......................................................

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- Textbook, notes, mobiles and smart devices are not allowed in this exam.
- Write neatly and legibly. You may lose points for messy work.
- Show all your work. No points for answers without justification.

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| Question | Marks | Max Marks |
|---|---|---|
| 1 | | 25 |
| 2 | | 15 |
| 3 | | 15 |
| 4 | | 20 |
| 5 | | 15 |
| 6 | | 10 |
| Total | | 100 |

1. Given the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 4 & 11 & 16 \end{bmatrix}$$

   (a) Find elementary matrices $E_1$, $E_2$, and $E_3$ such that $E_3E_2E_1A = U$, where $U$ is an upper triangular matrix.

   (b) Find the LU factorization of the matrix $A$ (i.e., demonstrate that $A = LU$).

   (c) Using the results from part (b), solve the system $Ax = d$ where $d = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.

2. Under what conditions on its entries is the matrix $B$ invertable?

$$B = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix}$$

3. Under what conditions on $b_1$ and $b_2$ (if any) does $Ax = b$ have a solution?

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

   Find the nullspace of $A$, and the complete solution to $Ax = b$.

4. Consider the following $3 \times 5$ matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -3 & -7 & -2 \end{bmatrix}$$

   Find the reduced row echelon form $R$, and write down the bases for the four fundamental subspaces associated with $A$.

5. Solve **3** questions from the following:

   (a) Find a 1 by 3 matrix whose nullspace consists of all vectors in $\mathbb{R}^3$ such that $x_1 + 2x_2 + 4x_3 = 0$. Find a 3 by 3 matrix with the same nullspace.

   (b) If $Ax = b$ always has at least one solution, show that the only solution to $A^Ty = 0$ is $y = 0$.

   (c) If $AB = 0$, the columns of B are in the nullspace of $A$. If those vectors are in $\mathbb{R}^n$, prove that $rank(A) + rank(B) \leq n$.

   (d) If $u$ and $v$ are linearly independent vectors, show that $u + v$ and $u - v$ are linearly independent.

   (e) Show that the set of all positive real numbers, with $x + y$ and $cx$ redefined to equal the usual $xy$ and $x^c$, is a vector space. What is the "zero vector"?

6. Write down the 6 by 4 incidence matrix of the following graph: