King Fahd University of Petroleum and Minerals
College of Computing and Mathematics
Department of Mathematics

Math 432 - Major Exam II

AY 2023-2024 (Term 232)

Time Allowed: 120 Minutes

Name: .... **ID number:** ....

- Textbook, notes, mobiles and smart devices are not allowed in this exam.
- Write neatly and legibly. You may lose points for messy work.
- Show all your work. No points for answers without justification.

Question	**Marks**	**Max Marks**
1		20
2		15
3		10
4		15
5		20
6		20
Total		100

1. Given the matrix

$$A = \begin{bmatrix} 1 & -6 \\ 3 & 6 \\ 4 & 8 \\ 5 & 0 \\ 7 & 8 \end{bmatrix}$$

   (a) Find an orthonormal basis for the column space of A.

   (b) Write A as QR, where Q has orthonormal columns and R is upper triangular.

   (c) Find the least-squares solution to $Ax = b$, if $b = (-3, 7, 1, 0, 4)^T$.

2. Find the straight line $C + Dt$ that best fits the measurements $b = 0, 1, 2, 5$ at times $t = 0, 1, 3, 4$.

3. Construct the projection matrix P onto the space spanned by $(1, 1, 1)$ and $(0, 1, 3)$.

4. If the eigenvalues of A are $0, 1, 2$, which of the following are certain to be true? Give a reason if true or a counterexample if false: (a) A is not invertible. (b) A is diagonalizable. (c) A is not diagonalizable.

5. Suppose there is an epidemic in which every month half of those who are well become sick, and a quarter of those who are sick recover. The transitions between these states can be represented using the following system:

$$\begin{bmatrix} R_{k+1} \\ S_{k+1} \\ W_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} R_k \\ S_k \\ W_k \end{bmatrix}$$

   where W_k, S_k, R_k are the number of well, sick, and recovered people at month k, respectively.

   (a) Verify that the transition process described above constitutes a Markov process.

   (b) Given the initial state vector $(0, 0, 2^8)^T$, find the distribution of the populations after 4 months.

   (c) Find the steady state of the system.

6. Suppose the rabbit population r and the wolf population w are governed by the following differential equations:

$$\frac{dr}{dt} = 4r - 2w,$$
$$\frac{dw}{dt} = r + w.$$

   (a) Is this system stable, neutrally stable, or unstable?

   (b) If initially $r = 300$ and $w = 200$, what are the populations at time t?

   (c) After a long time, what is the proportion of rabbits to wolves?